ON DEGENERATE PARTIAL DIFFERENTIAL EQUATIONS

GUI-QIANG G. CHEN

Abstract. Some of recent developments, including recent results, ideas, techniques, and approaches, in the study of degenerate partial differential equations are surveyed and analyzed. Several examples of nonlinear degenerate, even mixed, partial differential equations, are presented, which arise naturally in some longstanding, fundamental problems in fluid mechanics and differential geometry. The solution to these fundamental problems greatly requires a deep understanding of nonlinear degenerate partial differential equations. Our emphasis is on exploring and/or developing unified mathematical approaches, as well as new ideas and techniques. The potential approaches we have identified and/or developed through these examples include kinetic approaches, free boundary approaches, weak convergence approaches, and related nonlinear ideas and techniques. We remark that most of the important problems for nonlinear degenerate partial differential equations are truly challenging and still widely open, which require further new ideas, techniques, and approaches, and deserve our special attention and further efforts.

1. Introduction

We survey and analyze some of recent developments, including recent results, ideas, techniques, and approaches, in the study of degenerate partial differential equations. We start with several important examples of degenerate/mixed linear degenerate equations and some of their interrelations. Then we present several examples of nonlinear degenerate, even mixed, partial differential equations, arising naturally in some longstanding, fundamental problems in fluid mechanics and differential geometry. These examples indicate that some of important nonlinear degenerate problems are ready to be tractable. Our emphasis is on exploring and/or developing unified mathematical approaches, as well as new ideas and techniques. The potential approaches we have identified and/or developed through these examples include kinetic approaches, free boundary approaches, weak convergence approaches, and related nonlinear ideas and techniques.

In fact, nonlinear degenerate, even mixed, partial differential equations arise also naturally in fundamental problems in many other areas such as elasticity, relativity, optimization, dynamical systems, complex analysis, and string theory. The solution to these fundamental problems in the areas greatly requires a deep understanding of nonlinear

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degenerate partial differential equations. On the other hand, most of the important problems for nonlinear degenerate partial differential equations are truly challenging and still widely open, which require further new ideas, techniques, and approaches, and deserve our special attention and further efforts.

During the last half century, three different types of nonlinear partial differential equations (elliptic, hyperbolic, parabolic) have been systematically studied separately, and great progress has been made through the efforts of several generations of mathematicians. With these achievements, it is the time to revisit and attack nonlinear degenerate, even mixed, partial differential equations with emphasis on exploring and developing unified mathematical approaches, as well as new ideas and techniques.

The organization of this paper is as follows. In Section 2, we present several important examples of linear degenerate, even mixed, equations and exhibit some of their interrelations. In Section 3, we first reveal a natural connection between degenerate hyperbolic systems of conservation laws and the Euler-Poisson-Darboux equation through the entropy and Young measure, and then we discuss how this connection can be applied to solving hyperbolic systems of conservation laws with parabolic or hyperbolic degeneracy. In Section 4, we present a kinetic approach to handle a class of nonlinear degenerate parabolic-hyperbolic equations, the anisotropic degenerate diffusion-advection equation. In Section 5, we present two approaches through several examples to handle nonlinear mixed problems, especially nonlinear degenerate elliptic problems: free-boundary techniques and weak convergence methods. In Section 6, we discuss how the singular limits to nonlinear degenerate hyperbolic systems of conservation laws via weak convergence methods can be achieved through an important limit problem: the vanishing viscosity limit problem for the Navier-Stokes equations to the isentropic Euler equations.

2. Linear Degenerate Equations

In this section, we present several important examples of linear degenerate, even mixed, equations and exhibit some of their interrelations.

Three of the most basic types of partial differential equations are elliptic, parabolic, and hyperbolic. Consider the partial differential equations of second-order:

$$\sum_{i,j=1}^{d} a_{ij}(\mathbf{x})\partial_{x_ix_j}u+\sum_{j=1}^{d} b_j(\mathbf{x})\partial_{x_j}u+c(\mathbf{x})u=f(\mathbf{x}), \quad \mathbf{x}=(x_1,x_2,\cdots,x_d)\in\Omega\subset\mathbb{R}^d, \tag{2.1}$$

where $a_{ij}(\mathbf{x}), b_j(\mathbf{x}), c(\mathbf{x})$, and $f(\mathbf{x})$ are bounded for $\mathbf{x}\in\Omega$. Equation (2.1) is called a uniformly elliptic equation in Ω provided that there exists $\lambda_0>0$ such that, for any $\boldsymbol{\xi}=(\xi_1,\cdots,\xi_d)\in\mathbb{R}^d$,

$$\sum_{i,j=1}^{d} a_{ij}(\mathbf{x})\xi_i\xi_j\ge\lambda_0|\boldsymbol{\xi}|^2 \qquad \text{for } \mathbf{x}\in\Omega, \tag{2.2}$$

that is, the $d\times d$ matrix $(a_{ij}(\mathbf{x}))$ is positive definite.

Two of the basic types of time-dependent partial differential equations of second-order are hyperbolic equations:

$$\partial_{tt}u-L_{\mathbf{x}}u=f(t,\mathbf{x}), \qquad t>0, \tag{2.3}$$

and parabolic equations:

$$\partial_t u - L_{\mathbf{x}} u = f(t,\mathbf{x}), \qquad t > 0, \tag{2.4}$$

where $L_{\mathbf{x}}$ is a second-order elliptic operator:

$$L_{\mathbf{x}} u = \sum_{i,j=1}^{d} a_{ij}(t,\mathbf{x})\partial_{x_i x_j} u + \sum_{j=1}^{d} b_j(t,\mathbf{x})\partial_{x_j} u + c(t,\mathbf{x})u, \tag{2.5}$$

for which $a_{ij}(t,\mathbf{x}), b_j(t,\mathbf{x}), c(t,\mathbf{x})$, and $f(t,\mathbf{x})$ are locally bounded for $(t,\mathbf{x}) \in \mathbb{R}^{d+1}_+ := [0,\infty) \times \mathbb{R}^d$. Equations (2.3) and (2.4) are called uniformly hyperbolic or parabolic equations, respectively, in a domain in $\mathbb{R}^{d+1}_+$ under consideration, provided that there exists $\lambda_0 > 0$ such that, for any $\boldsymbol{\xi} \in \mathbb{R}^d$,

$$\sum_{i,j=1}^{d} a_{ij}(t,\mathbf{x})\xi_i\xi_j \geq \lambda_0 |\boldsymbol{\xi}|^2 \tag{2.6}$$

on the domain, that is, the $d \times d$ matrix $(a_{ij}(t,\mathbf{x}))$ is positive definite. In particular, when the coefficient functions a_{ij}, b_j, c, and the nonhomogeneous term f are time-invariant, and the solution u is also time-invariant, then equations (2.3) and (2.4) are coincide with equation (2.1).

Their representatives are Laplace's equation:

$$\Delta_{\mathbf{x}} u = 0, \tag{2.7}$$

the wave equation:

$$\partial_{tt} u - \Delta_{\mathbf{x}} u = 0, \tag{2.8}$$

and the heat equation:

$$\partial_t u - \Delta_{\mathbf{x}} u = 0, \tag{2.9}$$

respectively, where $\Delta_{\mathbf{x}} = \sum_{j=1}^{d} \partial_{x_j x_j}$ is the Laplace operator.

Similarly, a system of partial differential equations of first-order in one-dimension:

$$\partial_t U + \mathbf{A}(t,x)\partial_x U = 0, \qquad U \in \mathbb{R}^n \tag{2.10}$$

is called a strictly hyperbolic system, provided that the $n \times n$ matrix $\mathbf{A}(t,\mathbf{x})$ has n real, distinct eigenvalues:

$$\lambda_1(t,x) < \lambda_2(t,x) < \cdots < \lambda_n(t,x)$$

in the domain under consideration. The simplest example is

$$\mathbf{A}(t,x) = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}. \tag{2.11}$$

Then system (2.10)–(2.11) with $U = (u,v)^\top$ is equivalent to the two wave equations:

$$\partial_{tt} u - \partial_{xx} u = 0, \qquad \partial_{tt} v - \partial_{xx} v = 0.$$

However, many important partial differential equations are degenerate or mixed. That is, for the linear case, the matrix $(a_{ij}(t,\mathbf{x}))$ or $(a_{ij}(\mathbf{x}))$ is not positive definite or even indefinite, and the eigenvalues of the matrix $\mathbf{A}(t,x)$ are not distinct.

2.1. **Degenerate Equations and Mixed Equations.** Two prototypes of linear degenerate equations are the *Euler-Poisson-Darboux equation*:

$$(x-y)\partial_{xy}u+\alpha\partial_x u+\beta\partial_y u=0, \tag{2.12}$$

or

$$x(\partial_{xx}u-\partial_{yy}u)+\alpha\partial_x u+\beta\partial_y u=0, \tag{2.13}$$

and the *Beltrami equation*:

$$x(\partial_{xx}u+\partial_{yy}u)+\alpha\partial_x u+\beta\partial_y u=0. \tag{2.14}$$

Three prototypes of mixed hyperbolic-elliptic equations are the *Lavrentyev-Betsadze equation*:

$$\partial_{xx}u+\mathrm{sign}(x)\partial_{yy}u=0 \tag{2.15}$$

which exhibits the jump from the hyperbolic phase $x<0$ to the elliptic phase $x>0$, the *Tricomi equation*:

$$\partial_{xx}u+x\partial_{yy}u=0 \tag{2.16}$$

which exhibits hyperbolic degeneracy at $x=0$ (i.e., two eigenvalues coincide at $x=0$, but the corresponding characteristic curves are not tangential to the line $x=0$), and the *Keldysh equation*:

$$x\partial_{xx}u+\partial_{yy}u=0 \tag{2.17}$$

which exhibits parabolic degeneracy at $x=0$ (i.e., two eigenvalues coincide at $x=0$, but the corresponding characteristic curves are tangential to the line $x=0$).

The mixed parabolic-hyperbolic equations include the linear degenerate diffusion-advection equation:

$$\partial_t u+\mathbf{b}\cdot\nabla u=\nabla\cdot(\mathbf{A}\,\nabla u), \qquad \mathbf{x}\in\mathbb{R}^d,\ \ t\in\mathbb{R}_+:=(0,\infty), \tag{2.18}$$

where $u:\mathbb{R}_+\times\mathbb{R}^d\to\mathbb{R}$ is unknown, $\nabla=(\partial_{x_1},\dots,\partial_{x_d})$ is the gradient operator with respect to $\mathbf{x}=(x_1,\dots,x_d)\in\mathbb{R}^d$, and $\mathbf{b}:\mathbb{R}_+\times\mathbb{R}^d\to\mathbb{R}^d$ and $\mathbf{A}:\mathbb{R}_+\times\mathbb{R}^d\to\mathbb{R}^{d\times d}$ are given functions such that $\mathbf{A}=(a_{ij}(t,\mathbf{x}))$ is a $d\times d$ nonnegative, symmetric matrix. When the diffusion matrix function $\mathbf{A}$ degenerates, the advection term $\mathbf{b}\cdot\nabla u$ dominates; otherwise, the parabolic diffusion term $\nabla\cdot(\mathbf{A}\,\nabla u)$ dominates.

2.2. **Interrelations between the Linear Equations.** The above linear equations are not actually independent, but are closely interrelated.

2.2.1. *The Wave Equation* (2.8) *and the Euler-Poisson-Darboux Equation* (2.13). Seek spherically symmetric solutions of the wave equation (2.8):

$$v(t,r)=u(t,\mathbf{x}), \qquad r=|\mathbf{x}|.$$

Then $v(t,r)$ is governed by the Euler-Poisson-Darboux equation (2.13) with $\alpha=d-1$ and $\beta=0$:

$$\partial_{tt}v-\partial_{rr}v-\frac{d-1}{r}\partial_r v=0.$$

As is well-known, the Euler-Poisson-Darboux equation plays an important role in the spherical mean method for the wave equation to obtain the explicit representation of the solution in $\mathbb{R}^d, d\geq 2$ (cf. Evans [69]).

2.2.2. *The Tricomi Equation* (2.16)*, the Beltrami Equation* (2.14)*, and the Euler-Poisson-Darboux Equation* (2.13)*.* Under the coordinate transformations:

$$(x,y) \longrightarrow (\tau, y) = (\tfrac{2}{3}(-x)^{\frac{3}{2}}, y), \qquad \text{when } x < 0,$$
$$(x,y) \longrightarrow (\tau, y) = (\tfrac{2}{3}x^{\frac{3}{2}}, y), \qquad \text{when } x > 0,$$

the Tricomi equation (2.16) is transformed into the Beltrami equation (2.14) with $\alpha = \frac{1}{3}$ and $\beta = 0$ when $x = \tau > 0$:

$$\partial_{\tau\tau}u + \partial_{yy}u + \frac{1}{3\tau}\partial_\tau u = 0,$$

and the Euler-Poisson-Darboux equation (2.13) with $\alpha = \frac{1}{3}$ and $\beta = 0$ when $x = \tau < 0$:

$$\partial_{\tau\tau}u - \partial_{yy}u + \frac{1}{3\tau}\partial_\tau u = 0.$$

These show that a solution to one of them implies the solution of the other correspondingly, which are equivalent correspondingly.

Linear degenerate partial differential equations have been relatively better understood since 1950. The study of nonlinear partial differential equations has been focused mainly on the equations of single type during the last half century. The three different types of nonlinear partial differential equations have been systematically studied separately, and one of the main focuses has been on the tools, techniques, and approaches to understand different properties and features of solutions of the equations with these three different types. Great progress has been made through the efforts of several generations of mathematicians.

As we will see through several examples below, nonlinear degenerate, even mixed, partial differential equations naturally arise in some fundamental problems in fluid mechanics and differential geometry. The examples include nonlinear degenerate hyperbolic systems of conservation laws, the nonlinear degenerate diffusion-advection equation and the Euler equations for compressible flow in fluid mechanics, and the Gauss-Codazzi system for isometric embedding in differential geometry. Such degenerate, or mixed, equations naturally arise also in fundamental problems in many other areas including elasticity, relativity, optimization, dynamical systems, complex analysis, and string theory. The solution to these fundamental problems in the areas greatly requires a deep understanding of nonlinear degenerate, even mixed, partial differential equations.

3. Nonlinear Degenerate Hyperbolic Systems of Conservation Laws

Nonlinear hyperbolic systems of conservation laws in one-dimension take the following form:

$$\partial_t U + \partial_x F(U) = 0, \qquad U \in \mathbb{R}^n,\ (t,x) \in \mathbb{R}^2_+. \tag{3.1}$$

For any C^1 solutions, (3.1) is equivalent to

$$\partial_t U + \nabla F(U)\partial_x U = 0, \qquad U \in \mathbb{R}^n,\ (t,x) \in \mathbb{R}^2_+.$$

Such a system is hyperbolic if the $n \times n$ matrix $\nabla F(U)$ has n real eigenvalues $\lambda_j(U)$ and linearly independent eigenvectors $\mathbf{r}_j(U), 1 \le j \le n$. Denote

$$\mathcal{D} := \{U : \lambda_i(U) = \lambda_j(U), \quad i \neq j,\ 1 \le i,j \le n\} \tag{3.2}$$

as the degenerate set. If the set $\mathcal{D}$ is empty, then this system is strictly hyperbolic and, otherwise, degenerate hyperbolic. Such a set allows a degree of interaction, or nonlinear resonance, among different characteristic modes, which is missing in the strictly hyperbolic case but causes additional analytic difficulties. A point $U_* \in \mathcal{D}$ is hyperbolic degenerate if $\nabla F(U_*)$ is diagonalizable and, otherwise, is parabolic degenerate.

Degenerate hyperbolic systems of conservation laws have arisen from many important fields such as continuum mechanics including the vacuum problem, multiphase flows in porous media, MHD, and elasticity. On the other hand, degenerate hyperbolicity of systems is generic in some sense. For example, for three-dimensional hyperbolic systems of conservation laws, Lax [102] indicated that systems with $2 (mod\, 4)$ equations must be degenerate hyperbolic. The result is also true when the systems have $\pm 2, \pm 3, \pm 4 (mod\, 8)$ equations (see [75]). Then the plane wave solutions of such systems are governed by the corresponding one-dimensional hyperbolic systems with $\mathcal{D} \neq \emptyset$.

Since F is a nonlinear function, solutions of the Cauchy problem for (3.1) (even starting from smooth initial data) generally develop singularities in a finite time, and then the solutions become discontinuous functions. This situation reflects in part the physical phenomenon of breaking of waves and development of shock waves. For this reason, attention is focused on solutions in the space of discontinuous functions, where one can not directly use the classical analytic techniques that predominate in the theory of partial differential equations of other types.

To overcome this difficulty, a natural idea is to construct approximate solutions $U^\epsilon(t,x)$ to (3.1) by using shock capturing methods and then to study the convergence and consistency of the approximate solutions to (3.1). The key issue is whether the approximate solutions converge in an appropriate topology and the limit function is consistent with (3.1). Solving this issue involves two aspects: one is to construct good approximate solutions, and the other is to make suitable compactness analysis in an appropriate topology.

3.1. **Connection with the Euler-Poisson-Darboux Equation: Entropy and Young Measure.**

The connection between degenerate hyperbolic systems of conservation laws and the Euler-Poisson-Darboux equation is through the entropy.

A pair of functions $(\eta, q) : \mathbb{R}^n \to \mathbb{R}^2$ is called an entropy-entropy flux pair (entropy pair, in short) if they satisfy the following linear hyperbolic system:

$$\nabla q(U) = \nabla \eta(U) \nabla F(U). \tag{3.3}$$

Then the function $\eta(U)$ is called an entropy. Clearly, any C^1 solution satisfies

$$\partial_t \eta(U) + \nabla \eta(U) \nabla F(U) \partial_x U = 0, \tag{3.4}$$

or

$$\partial_t \eta(U) + \partial_x q(U) = 0. \tag{3.5}$$

For a BV solution that is not C^1, the second term in (3.4) has no meaning in the classical sense because of the multiplication of a Radon measure with a discontinuous function. If $\eta(U)$ is an entropy, then the left side of (3.4) becomes the left side of (3.5) that makes sense even for L^∞ solutions in the sense of distributions. An L^∞ function $U(t,x)$ is called an entropy solution (cf. Lax [101]) if

$$\partial_t \eta(U) + \partial_x q(U) \leq 0 \tag{3.6}$$

in the sense of distributions for any convex entropy pair, that is, the Hessian $\nabla^2 \eta(U) \geq 0$.

Assume that system (3.1) is endowed with globally defined Riemann invariants $\mathbf{w}(U) = (w_1, \ldots, w_n)(U), 1 \leq i \leq n$, which satisfy

$$\nabla w_i(U) \cdot \nabla F(U) = \lambda_i(U) \nabla w_i(U). \tag{3.7}$$

The necessary and sufficient condition for the existence of the Riemann invariants $w_i(U), 1 \leq i \leq n$, for strictly hyperbolic systems is the well-known Frobenius condition:

$$\mathbf{l}_i\{\mathbf{r}_j, \mathbf{r}_k\} = 0, \qquad \text{for any } \ j, k \neq i,$$

where $\mathbf{l}_i$ denotes the left eigenvector corresponding to λ_i and $\{\cdot, \cdot\}$ is the Poisson bracket of vector fields in the U-space (cf. [1, 99]). For $n = 2$, the Frobenius condition always holds. For any smooth solution $U(t, x)$, the corresponding Riemann invariants $w_i, 1 \leq i \leq n$, satisfy the transport equations

$$\partial_t w_i + \lambda_i(\mathbf{w}) \partial_x w_i = 0,$$

which indicate that w_i is invariant along the i-th characteristic field.

Taking the inner product of (3.7) with the right eigenvectors $\mathbf{r}_i$ of ∇F produces the characteristic form

$$(\lambda_i \nabla \eta - \nabla q) \cdot \mathbf{r}_i = 0,$$

that is,

$$\lambda_i \partial_{w_i} \eta = \partial_{w_i} q. \tag{3.8}$$

For $n = 2$, the linear hyperbolic system (3.7) is equivalent to (3.8), and system (3.8) can be reduced to the following linear second-order hyperbolic equation:

$$\partial_{w_1 w_2} \eta + \frac{\partial_{w_1} \lambda_2}{\lambda_2 - \lambda_1} \partial_{w_2} \eta - \frac{\partial_{w_2} \lambda_1}{\lambda_2 - \lambda_1} \partial_{w_1} \eta = 0. \tag{3.9}$$

For the case $\mathcal{D} = \emptyset$, equation (3.9) is, in general, regular and hyperbolic for which either the Goursat problem or the Cauchy problem is well posed in the coordinates of Riemann invariants. The entropy space is infinite-dimensional and is represented by two families of functions of one variable. However, for the case $\mathcal{D} \neq \emptyset$, the situation is much more complicated because of the singularity of the functions $\frac{\partial_{w_1} \lambda_2}{\lambda_2 - \lambda_1}$ and $\frac{\partial_{w_2} \lambda_1}{\lambda_2 - \lambda_1}$ on the set $\mathbf{w}(\mathcal{D}) \subset \mathbb{R}^2$. The typical form of such equations is

$$\partial_{w_1 w_2} \eta + \frac{\alpha(\frac{w_1}{w_2}, w_1 - w_2)}{w_1 - w_2} \partial_{w_1} \eta + \frac{\beta(\frac{w_1}{w_2}, w_1 - w_2)}{w_1 - w_2} \partial_{w_2} \eta = 0. \tag{3.10}$$

This equation may have extra singularities of α and β both at the origin and on the line $w_1 = w_2$. The questions are whether there exist nontrivial regular solutions of the singular equation (3.10) and, if so, how large the set of smooth regular solutions is.

The connection between the compactness problem of approximate solutions and the entropy determined by the Euler-Poisson-Darboux equation (3.9) is the Young measure via the compensated compactness ideas first developed by Tartar [147, 148] and Murat [119, 120] and a related observation presented by Ball [7].

The Young measure is a useful tool for studying the limiting behavior of measurable function sequences. For an arbitrary sequence of measurable maps

$$U^\epsilon : \mathbb{R}^2_+ \to \mathbb{R}^n, \quad \|U^\epsilon\|_{L^\infty} \leq C < \infty,$$

that converges in the weak-star topology of L^∞ to a function U,

$$w^* - \lim_{\epsilon\to 0} U^\epsilon = U,$$

there exist a subsequence (still labeled) U^ϵ and a family of Young measures

$$\nu_{t,x} \in \text{Prob.}(\mathbb{R}^n), \quad \text{supp}\ \nu_{t,x} \subset \{\boldsymbol{\lambda} : |\boldsymbol{\lambda}| \le M\}, \qquad (t,x) \in \mathbb{R}^2_+,$$

such that, for any continuous function g,

$$w^* - \lim_{\epsilon\to 0} g(U^\epsilon(t,x)) = \int_{\mathbb{R}^n} g(\boldsymbol{\lambda}) d\nu_{t,x}(\boldsymbol{\lambda}) := \langle \nu_{t,x}, g\rangle$$

for almost all points $(t,x) \in \mathbb{R}^2_+$. In particular, $U^\epsilon(t,x)$ converges strongly to $U(t,x)$ if and only if the Young measure $\nu_{t,x}$ is equal to a Dirac mass concentrated at $U(t,x)$ for a.e. (t,x).

In many cases, one can estimate (cf. [61, 30]) that the approximate solutions $U^\epsilon(t,x)$ generated by the shock capturing methods for (3.3) satisfy

- $\|U^\epsilon\|_{L^\infty} \le C < \infty$.
- For C^2 entropy pairs $(\eta_i, q_i), i = 1, 2$, determined by (3.8) and (3.9),

$$\partial_t \eta_i(U^\epsilon) + \partial_x q_i(U^\epsilon) \quad \text{compact in } H^{-1}_{\text{loc}}. \tag{3.11}$$

Then, for any C^2 entropy pairs $(\eta_i, q_i), i = 1, 2$, determined by (3.8) and (3.9), the Young measure $\nu_{t,x}$ is forced to satisfy the Tartar-Murat commutator relation:

$$\langle \nu_{t,x}, \eta_1 q_2 - q_1 \eta_2\rangle = \langle \nu_{t,x}, \eta_1\rangle\langle \nu_{t,x}, q_2\rangle - \langle \nu_{t,x}, q_1\rangle\langle \nu_{t,x}, \eta_2\rangle \qquad a.e.\ (t,x) \in \mathbb{R}^2_+. \tag{3.12}$$

This relation is derived from (3.11), the Young measure representation theorem for the measurable function sequence, and a basic continuity theorem for the 2×2 determinant in the weak topology (cf. [147, 30]). This indicates that proving the compactness of the approximate solutions $U^\epsilon(t,x)$ is equivalent to solving the functional equation (3.12) for the Young measure $\nu_{t,x}$ for all possible C^2 entropy pairs $(\eta_i, q_i), i = 1, 2$, determined by (3.8) and (3.9). If one clarifies the structure of the Young measure $\nu_{t,x}(\boldsymbol{\lambda})$, one can understand the limiting behavior of the approximate solutions. For example, if one can prove that almost all Young measures $\nu_{t,x}, (t,x) \in \mathbb{R}^2_+$, are Dirac masses, then one can conclude the strong convergence of the approximate solutions $U^\epsilon(t,x)$ almost everywhere. On the other hand, the structure of the Young measures is determined by the C^2 solutions of the Euler-Poisson-Darboux equation (3.9) via (3.12). One of the principal difficulties for the reduction is the general lack of enough classes of entropy functions that can be verified to satisfy certain weak compactness conditions. This is due to possible singularities of entropy functions near the regions with degenerate hyperbolicity. The larger the set of C^2 solutions to the entropy equation (3.9) of the Euler-Poisson-Darboux type is, the easier solving the functional equation (3.12) for the Young measure is.

3.2. **Hyperbolic Conservation Laws with Parabolic Degeneracy.** One of the prototypes for such systems is the system of isentropic Euler equations with the form

$$\begin{cases} \partial_t \rho + \partial_x m = 0, \\ \partial_t m + \partial_x(\frac{m^2}{\rho} + p(\rho)) = 0, \end{cases} \tag{3.13}$$

where ρ, m, and p are the density, mass, and pressure, respectively. For $\rho > 0$, $v = m/\rho$ represents the velocity of the fluid. The physical region for (3.13) is $\{(\rho, m) : |m| \le C\rho\}$

for some $C > 0$, in which the term $\frac{m^2}{\rho}$ in the flux function is only at most Lipschitz continuous near the vacuum. For $\rho > 0$, $v = \frac{m}{\rho}$ represents the velocity of the fluid.

The pressure p is a function of the density through the internal energy $e(\rho)$:

$$p(\rho) = \rho^2 e'(\rho) \qquad \text{for } \rho \geq 0. \tag{3.14}$$

In particular, for a polytropic perfect gas,

$$p(\rho) = \kappa\rho^\gamma, \qquad e(\rho) = \int_0^\rho \frac{p(r)}{r^2} dr = \frac{\kappa}{\gamma - 1}\rho^{\gamma-1}, \tag{3.15}$$

where $\gamma > 1$ is the adiabatic exponent and, by the scaling, the constant κ in the pressure-density relation may be chosen as $\kappa = \frac{(\gamma-1)^2}{4\gamma}$ without loss of generality.

For (3.13), strict hyperbolicity and genuine nonlinearity away from the vacuum require that

$$p'(\rho) > 0, \quad 2\,p'(\rho) + \rho\, p''(\rho) > 0 \qquad \text{for } \ \rho > 0. \tag{3.16}$$

Near the vacuum, for some $\gamma > 1$,

$$\frac{p(\rho)}{\rho^\gamma} \to \kappa_1 > 0 \qquad \text{when } \rho \to 0. \tag{3.17}$$

One of the fundamental features of this system is that strict hyperbolicity fails when $\rho \to 0$. That is, $\mathcal{D} = \{\rho = 0\}$, which is the vacuum state, and the degeneracy is parabolic.

The eigenvalues of (3.13) are

$$\lambda_1 = \frac{m}{\rho} - c(\rho), \qquad \lambda_2 = \frac{m}{\rho} + c(\rho),$$

and the corresponding Riemann invariants of (3.13) are

$$w_1 = \frac{m}{\rho} - k(\rho), \qquad w_2 = \frac{m}{\rho} + k(\rho),$$

where

$$c(\rho) := \sqrt{p'(\rho)}$$

is the sound speed, and

$$k(\rho) := \int_0^\rho \frac{c(r)}{r}\, dr.$$

Then the entropy equation (3.9) in the Riemann coordinates is the following Euler-Poisson-Darboux equation:

$$\partial_{w_1 w_2}\eta + \frac{\alpha(w_1 - w_2)}{w_1 - w_2}(\partial_{w_1}\eta - \partial_{w_2}\eta) = 0, \tag{3.18}$$

where

$$\alpha(w_1 - w_2) = -\frac{k(\rho)k''(\rho)}{k'(\rho)^2}$$

with $\rho = -k^{-1}(\frac{w_1 - w_2}{2})$.

For the γ-law gas,

$$\alpha(w_1 - w_2) = \lambda := \frac{3 - \gamma}{2(\gamma - 1)}$$

is a constant, the simplest case.

In terms of the variables (ρ, v), η is determined by the following second-order linear wave equation:

$$\partial_{\rho\rho}\eta - k'(\rho)^2\,\partial_{vv}\eta = 0. \tag{3.19}$$

An entropy η is called a weak entropy if $\eta(0,v)=0$. For example, the mechanical energy pair:

$$\eta_* = \frac{1}{2}\frac{m^2}{\rho} + \rho e(\rho), \quad q_* = \frac{m^3}{2\rho^2} + m\big(e(\rho) + \frac{p(\rho)}{\rho}\big),$$

is a convex weak entropy pair.

By definition, the weak *entropy kernel* is the solution $\chi(\rho, v, s)$ of the singular Cauchy problem

$$\begin{cases} \partial_{\rho\rho}\chi - k'(\rho)^2\,\partial_{vv}\chi = 0, \\ \chi(0, v, s) = 0, \\ \partial_\rho\chi(0, v, s) = \delta_{v=s} \end{cases} \tag{3.20}$$

in the sense of distributions, where s plays the role of a parameter and $\delta_{v=s}$ denotes the Dirac measure at $v = s$. Then the family of weak entropy functions is described by

$$\eta(\rho, v) = \int_{\mathbb{R}} \chi(\rho, v, s)\,\psi(s)\,ds, \tag{3.21}$$

where $\psi(v)$ is an arbitrary function. By construction, $\eta(0,v) = 0$, $\eta_\rho(0,v) = \psi(v)$. One can prove that, for $0 \le \rho \le C$, $|\frac{m}{\rho}| \le C$,

$$|\nabla\eta(\rho,m)| \le C_\eta, \qquad |\nabla^2\eta(\rho,m)| \le C_\eta \nabla^2\eta_*(\rho,m),$$

for any weak entropy η, with C_η independent of (ρ, m).

Since this system is a prototype in mathematical fluid dynamics, the mathematical study of this system has an extensive history dating back to the work of Riemann [134], where a special Cauchy problem, so-called Riemann problem, was solved. Zhang-Guo [157] established an existence theorem of global solutions to this system for a class of initial value functions by using the characteristic method. Nishida [121] obtained the first large data existence theorem with locally finite total-variation for the case $\gamma = 1$ using Glimm's scheme [80]. Large-data theorems have also been obtained for general $\gamma > 1$ in the case where the initial value functions with locally finite total-variation are restricted to prevent the development of cavities (e.g. [60, 108, 122]). The difficult point in bounding the total-variation norm at low densities is that the coupling between the characteristic fields increases as the density decreases. This difficulty is a reflection of the fact that the strict hyperbolicity fails at the vacuum: $\mathcal{D} \neq \emptyset$.

The first global existence for (3.13) with large initial data in L^∞ was established in DiPerna [61] for the case $\gamma = \frac{N+2}{N}$, $N \ge 5$ odd, by the vanishing viscosity method. The existence problem for usual gases with general values $\gamma \in (1, \frac{5}{3}]$ was solved in Chen [29] and Ding-Chen-Luo [59] (also see [30]). The case $\gamma \ge 3$ was treated by Lions-Perthame-Tadmor [107]. Lions-Perthame-Souganidis [106] dealt with the interval $(\frac{5}{3}, 3)$ and simplified the proof for the whole interval. A compactness framework has been established even for the general pressure law in Chen-LeFloch [44] by using only weak entropy pairs.

More precisely, assume the pressure function $p = p(\rho) \in C^4(0,\infty)$ satisfies condition (3.16) (i.e., strict hyperbolicity and genuine nonlinearity) away from the vacuum and, near

the vacuum, $p(\rho)$ is only asymptotic to the γ-law pressure (as real gases): there exist a sequence of exponents

$$1 < \gamma := \gamma_1 < \gamma_2 < \ldots < \gamma_N \leq (3\gamma - 1)/2 < \gamma_{N+1} \tag{3.22}$$

and a sufficiently smooth function $P = P(\rho)$ such that

$$p(\rho) = \sum_{j=1}^{N} \kappa_j \, \rho^{\gamma_j} + \rho^{\gamma_{N+1}} \, P(\rho), \tag{3.23}$$

$$P(\rho) \text{ and } \rho^3 \, P'''(\rho) \text{ are bounded as } \rho \to 0, \tag{3.24}$$

for some coefficients $\kappa_j \in \mathbb{R}$ with $\kappa_1 > 0$. The solutions under consideration will remain in a bounded subset of $\{\rho \geq 0\}$ so that the behavior of $p(\rho)$ for large ρ is irrelevant. This means that the pressure law $p(\rho)$ has the same singularity as $\sum_{j=1}^{N} \kappa_j \, \rho^{\gamma_j}$ near the vacuum. Observe that $p(0) = p'(0) = 0$, but, for $k > \gamma_1$, the higher derivative $p^{(k)}(\rho)$ is unbounded near the vacuum with different orders of singularity.

Theorem 3.1 (Chen-LeFloch [44]). *Consider system* (3.13) *with general pressure law satisfying* (3.16) *and* (3.22)–(3.24). *Assume that a sequence of functions* $(\rho^\varepsilon, m^\varepsilon)$ *satisfies that*

(i) *There exists* $C > 0$ *independent of* ε *such that*

$$0 \leq \rho^\varepsilon(t,x) \leq C, \qquad |m^\varepsilon(t,x)| \leq C\, \rho^\varepsilon(t,x) \qquad \text{for a.e. } (t,x); \tag{3.25}$$

(ii) *For any weak entropy pair* (η, q) *of* (3.13), (3.16), *and* (3.22)–(3.24),

$$\partial_t \eta(\rho^\varepsilon, m^\varepsilon) + \partial_x q(\rho^\varepsilon, m^\varepsilon) \quad \text{is compact in } H^{-1}_{loc}(\mathbb{R}^2_+). \tag{3.26}$$

Then the sequence $(\rho^\varepsilon, m^\varepsilon)$ *is compact in* $L^1_{loc}(\mathbb{R}^2_+)$. *Moreover, there exists a global entropy solution* $(\rho(t,x), m(t,x))$ *of the Cauchy problem* (3.13), (3.16), *and* (3.22)–(3.24), *satisfying*

$$0 \leq \rho(t,x) \leq C, \qquad |m(t,x)| \leq C\rho(t,x)$$

for some C *depending only on* C_0 *and* γ, *and*

$$\partial_t \eta(\rho, m) + \partial_x q(\rho, m) \leq 0$$

in the sense of distributions for any convex weak entropy pair (η, q). *Furthermore, the bounded solution operator* $(\rho, m)(t, \cdot) = S_t(\rho_0, m_0)(\cdot)$ *is compact in* L^1 *for* $t > 0$.

As discussed in Section 3.1, if the Young measure satisfying (3.12) reduces to a Dirac mass for a.e. (t,x), then the sequence $(\rho^\varepsilon, m^\varepsilon)$ is compact in the strong topology and converges subsequentially toward an entropy solution. For the Euler equations, to obtain that the Young measure $\nu_{(t,x)}$ is a Dirac mass in the (ρ, m)-plane, it suffices to prove that the measure in the (ρ, v)-plane, still denoted by $\nu_{(t,x)}$, is either a single point or a subset of the vacuum line

$$\{(\rho, v) \,:\, \rho = 0,\, |v| \leq \sup_{\varepsilon > 0} \big\| \frac{m^\varepsilon}{\rho^\varepsilon} \big\|_{L^\infty} \}.$$

The main difficulty is that only *weak* entropy pairs can be used, because of the vacuum problem.

In the proof of [29, 59, 61] (also cf. [30]), the heart of the matter is to construct special functions ψ in (3.21) in order to exploit the form of the set of constraints (3.12). These test-functions are suitable approximations of high-order derivatives of the Dirac measure.

It is used that (3.12) represents an *imbalance of regularity:* the operator on the left is more regular than the one on the right due to cancellation. DiPerna [61] considered the case that $\lambda := \frac{3-\gamma}{2(\gamma-1)} \geq 3$ is an integer so that the weak entropies are polynomial functions of the Riemann invariants. The new idea of applying the technique of fractional derivatives was first introduced in Chen [29] and Ding-Chen-Luo [59] to deal with real values of λ.

A new technique for equation (3.12) was introduced by Lions-Perthame-Tadmor [107] for $\gamma \in [3, \infty)$ and by Lions-Perthame-Souganidis [106] for $\gamma \in (1,3)$. Motivated by a kinetic formulation, they made the observation that the use of the test-functions ψ could in fact be bypassed, and (3.12) be directly expressed with the entropy kernel χ_*. Namely, (3.12) holds for all s_1 and s_2 by replacing $\eta_j := \chi_*(s_j)$ and $q_j := \sigma_*(s_j)$ for $j = 1, 2$. Here σ_* is the entropy-flux kernel defined as

$$\sigma_*(\rho, v, s) = \big(v + \theta\,(s-v)\big)\,\chi_*(\rho, v, s).$$

In [106], the commutator relations were differentiated in s, using the technique of fractional derivatives as introduced in [29, 59] by performing the operator $\partial_s^{\lambda+1}$, so that singularities arise by differentiation of χ_*. This approach relies again on the lack of balance in regularity of the two sides of (3.12) and on the observation that $\langle \nu_{(t,x)}, \chi_*(s)\rangle$ is smoother than the kernel $\chi_*(s)$ itself, due to the averaging by the Young measure.

However, many of the previous arguments do not carry over to the general pressure law. The main issue is to construct all of the weak entropy pairs of (3.13). The proof in Chen-LeFloch [44] has been based on the existence and regularity of the *entropy kernel* that generates the family of weak entropies via solving a new Euler-Poisson-Darboux equation, which is *highly singular* when the density of the fluid vanishes. New properties of *cancellation of singularities* in combinations of the entropy kernel and the associated entropy-flux kernel have been found. In particular, a new *multiple-term* expansion has been introduced based on the Bessel functions with suitable exponents, and the optimal assumption required on the singular behavior on the pressure law at the vacuum has been identified in order to valid the multiple-term expansion and to establish the existence theory. The results cover, as a special example, the density-pressure law $p(\rho) = \kappa_1\,\rho^{\gamma_1} + \kappa_2\,\rho^{\gamma_2}$ where $\gamma_1, \gamma_2 \in (1,3)$ and $\kappa_1, \kappa_2 > 0$ are arbitrary constants. The proof of the reduction theorem for Young measure has also further simplified the proof known for the polytropic perfect gas.

Then this compactness framework has been successfully applied to proving the convergence of the Lax-Friedrichs scheme, the Godunov scheme, and the artificial viscosity method for the isentropic Euler equations with the general pressure law.

3.3. **Hyperbolic Conservation Laws with Hyperbolic Degeneracy.** One of the prototypes of hyperbolic conservation laws with hyperbolic degeneracy is the gradient quadratic flux system, which is umbilic degeneracy, the most singular case:

$$\partial_t U + \partial_x(\nabla_U C(U)) = 0, \quad U = (u, v)^\top \in \mathbb{R}^2, \tag{3.27}$$

where

$$C(U) = \frac{1}{3}au^3 + bu^2v + uv^2, \tag{3.28}$$

and a and b are two real parameters.

Such systems are generic in the following sense. For any smooth nonlinear flux function, take its Taylor expansion about the isolated umibilic point. The first three terms including

the quadratic terms determine the local behavior of the hyperbolic singularity near the umbilic point. The hyperbolic degeneracy enables us to eliminate the linear term by a coordinate transformation to obtain the system with a homogeneous quadratic polynomial flux. Such a polynomial flux contains some inessential scaling parameters. There is a nonsingular linear coordinate transformation to transform the above system into (3.27)–(3.28), first studied by Marchesin, Isaacson, Plohr, and Temple, and in a more satisfactory form by Schaeffer-Shearer [135]. From the viewpoint of group theory, such a reduction from six to two parameters is natural: For the six dimensional space of quadratic mappings acted by the four dimensional group $GL(2,\mathbb{R})$, one expects the generic orbit to have codimension two.

The Riemann solutions for such systems were discussed by Isaacson, Marchesin, Paes-Leme, Plohr, Schaeffer, Shearer, Temple, and others (cf. [88, 89, 136, 140]). Two new types of shock waves, the overcompressive shock and the undercompressive shock, were discovered, which are quite different from the gas dynamical shock. The overcompressive shock can be easily understood using the Lax entropy condition [101]. It is known that there is a traveling wave solution connecting the left to right state of the undercompressive shock for the artificial viscosity system. Stability of such traveling waves for the overcompressive shock and the undercompressive shock has been studied (cf. [109, 110]).

The next issues are whether the compactness of the corresponding approximate solutions is affected by the viscosity matrix as the viscosity parameter goes to zero, to understand the sensitivity of the undercompressive shock with respect to the viscosity matrix, and whether the corresponding global existence of entropy solutions can be obtained as a corollary from this effort. The global existence of entropy solutions to the Cauchy problem for a special case of such quadratic flux systems was solved by Kan [93] via the viscosity method. A different proof was given independently to the same problem by Lu [111].

In Chen-Kan [41], an L^∞ compactness framework has been established for sequences of approximate solutions to general hyperbolic systems with umbilic degeneracy specially including (3.27)–(3.28). Under this framework, any approximate solution sequence, which is apriori uniformly bounded in L^∞ and produces the correct entropy dissipation, leads to the compactness of the corresponding Riemann invariant sequence. This means that the viscosity matrix does not affect the compactness of the corresponding uniformly bounded Riemann invariant sequence. Again, one of the principal difficulties associated with such systems is the general lack of enough classes of entropy functions that can be verified to satisfy certain weak compactness conditions. This is due to possible singularities of entropy functions near the regions of degenerate hyperbolicity. The analysis leading to the compactness involves two steps:

In the first step, we have constructed regular entropy functions governed by a highly singular entropy equation, the Euler-Poisson-Darboux type equation,

$$\partial_{w_1 w_2}\eta + \frac{\alpha(\frac{w_1}{w_2})}{w_2 - w_1}\partial_{w_1}\eta + \frac{\beta(\frac{w_1}{w_2})}{w_2 - w_1}\partial_{w_2}\eta = 0. \tag{3.29}$$

There are two main difficulties. The first is that the coefficients of the entropy equation are multiple-valued functions near the umbilic points in the Riemann invariant coordinates. This difficulty has been overcome by a detailed analysis of the singularities of the Riemann function of the entropy equation. This analysis involves a study of a corresponding Euler-Poisson-Darboux equation and requires very complicated estimates and calculations.

An appropriate choice of Goursat data leads to the cancellation of singularities and the achievement of regular entropies in the Riemann invariant coordinates. The second difficulty is that the nonlinear correspondence between the physical state coordinates and the Riemann invariant coordinates is, in general, irregular. A regular entropy function in the Riemann invariant coordinates is usually no longer regular in the physical coordinates. We have overcome this by a detailed analysis of the correspondence between these two coordinates.

In the second step, we have studied the structure of the Young measure associated with the approximate sequence and have proved that the support of the Young measures lies in finite isolated points or separate lines in the Riemann invariant coordinates. This has been achieved by a delicate use of Serre's technique [137] and regular entropy functions, constructed in the first step, in the Tartar-Murat commutator equation for Young measures associated with the approximate solution sequence.

This compactness framework has been successfully applied to proving the convergence of the Lax-Friedrichs scheme, the Godunov scheme, and the viscosity method for the quadratic flux systems. Some corresponding existence theorems of global entropy solutions for such systems have been established. The compactness has been achieved by reducing the support of the corresponding Young measures to a Dirac mass in the physical space.

4. Nonlinear Degenerate Parabolic-Hyperbolic Equations

One of the most important examples of nonlinear degenerate parabolic-hyperbolic equations is the nonlinear degenerate diffusion-advection equation of second-order with the form:

$$\partial_t u + \nabla \cdot \mathbf{f}(u) = \nabla \cdot (\mathbf{A}(u)\nabla u), \qquad \mathbf{x} \in \mathbb{R}^d, \quad t \in \mathbb{R}_+, \tag{4.1}$$

where $\mathbf{f} : \mathbb{R} \to \mathbb{R}^d$ satisfies $\mathbf{f}'(\cdot) \in L^\infty_{\text{loc}}(\mathbb{R}; \mathbb{R}^d)$, and the $d \times d$ matrix $\mathbf{A}(u)$ is symmetric, nonnegative, and locally bounded, so that $\mathbf{A}(u) = (\sigma_{ik}(u))(\sigma_{ik}(u))^\top$ with $\sigma_{ik}(u) \in L^\infty_{loc}(\mathbb{R})$. Equation (4.1) and its variants model anisotropic degenerate diffusion-advection motions of ideal fluids and arise in a wide variety of important applications, including two-phase flows in porous media and sedimentation-consolidation processes (cf. [19, 27, 67, 150]). Because of its importance in applications, there is a large literature for the design and analysis of various numerical methods to calculate the solutions of equation (4.1) and its variants (e.g. [27, 51, 65, 67, 91]) for which a well-posedness theory is in great demand.

One of the prototypes is the porous medium equation:

$$\partial_t u = \Delta_{\mathbf{x}}(u^m), \qquad m > 1, \tag{4.2}$$

which describes the fluid flow through porous media (cf. [150]). Equation (4.2) is degenerate on the level set $\{u = 0\}$; away from this set, i.e., on $\{u > 0\}$, the equation is strictly parabolic. Although equation (4.2) is of parabolic nature, the solutions exhibit certain hyperbolic feature, which results from the degeneracy. One striking family of solutions is Barenblatt's solutions found in [8]:

$$u(t, \mathbf{x}; a, \tau) = \frac{1}{(t+\tau)^k}\Big[a^2 - \frac{k(m-1)}{2md}\frac{|\mathbf{x}|^2}{(t+\tau)^{\frac{2k}{d}}}\Big]_+^{\frac{1}{m-1}},$$

where $k = \frac{d}{(m-1)d+2}$, and $a \neq 0$ and $\tau > 0$ are any constants. The dynamic boundary of the support of $u(t, \mathbf{x}; a, \tau)$ is

$$|\mathbf{x}| = \sqrt{\frac{2md}{k(m-1)}}a(t+\tau)^{\frac{k}{d}}, \qquad t \geq 0.$$

This shows that the support of Barenblatt's solutions propagates with a finite speed!

The simplest example for the isotropic case (i.e., $\mathbf{A}(u)$ is diagonal) with both phases is

$$\partial_t u + \partial_x(\frac{u^2}{2}) = \partial_{xx}[u]_+, \tag{4.3}$$

where $[u]_+ = \max\{u, 0\}$ (cf. [35]). Equation (4.3) is hyperbolic when $u < 0$ and parabolic when $u > 0$, and the level set $\{u = 0\}$ is a free boundary that is an interface separating the hyperbolic phase from the parabolic phase. For any constant states $u^\pm$ such that $u^+ < 0 < u^-$ and $y_* \in \mathbb{R}$, there exists a unique nonincreasing profile $\phi = \phi(y)$ such that

$$\begin{aligned}
&\lim_{y\to\pm\infty} \phi(y) = u^\pm, \qquad \lim_{y\to\pm\infty} \phi'(y) = 0,\\
&\phi(y) \equiv u^+ \qquad \text{for all } y > y_*,\\
&\phi \in C^2(-\infty, y_*), \;\; \phi(y_*-) = 0, \;\; \lim_{y\to y_*-} \phi'(y) = -\infty,\\
&\frac{d[\phi(y)]_+}{dy}\Big|_{y=y_*-} = \frac{1}{2}u^+u^-
\end{aligned}$$

so that $u(t, x) = \phi(x - st)$ is a discontinuous solution connecting u^+ from u^-, where $s = \frac{u^+ + u^-}{2}$. Although $u(t, x)$ is discontinuous, $[u(t, x)]_+$ is a continuous function even across the interface $\{u = 0\}$. See Fig. 1.

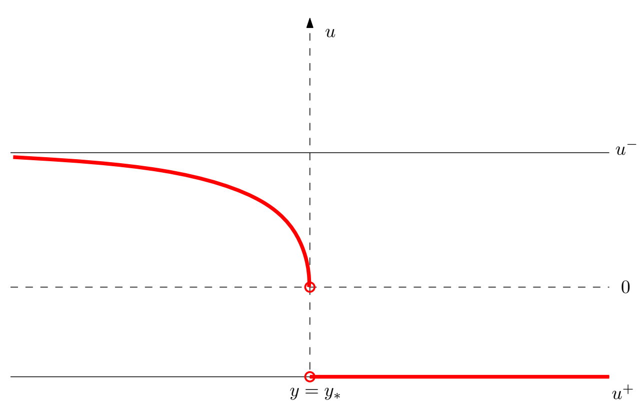

FIGURE 1. A discontinuous profile connecting the hyperbolic phase from the parabolic phase

The well-posedness issue for the Cauchy problem is relatively well understood if one removes the diffusion term $\nabla \cdot (\mathbf{A}(u)\nabla u)$, thereby obtaining a scalar hyperbolic conservation law; see Lax [100], Oleinik [123], Volpert [151], Kruzhkov [98], and Lions-Perthame-Tadmor [107], and Perthame [126] (also cf. [54, 143]). It is equally well understood for the diffusion-dominated case, especially when the set $\{u : rank(\mathbf{A}(u)) < d\}$ consists of only isolated

points with certain order of degeneracy; see Brezis-Crandall [17], Caffarelli-Friedman [21], Daskalopoulos-Hamilton [55], DiBenedetto [56], Gilding [79], Vázquez [150], and the references cited therein. For the isotropic diffusion, $a_{ij}(u) = 0, i \neq j$, some stability results for entropy solutions were obtained for BV solutions by Volpert-Hudjaev [152] in 1969. Only in 1999, Carrillo [23] extended this result to L^∞ solutions (also see Eymard-Gallouët-Herbin-Michel [72], Karlsen-Risebro [91] for further extensions), and Chen-DiBenedetto [35] handled the case of unbounded entropy solutions which may grow when $|\mathbf{x}|$ is large.

A unified approach, the kinetic approach, to deal with both parabolic and hyperbolic phases for the general anisotropic case with solutions in L^1 has been developed in Chen-Perthame [46]. This approach is motivated by the macroscopic closure procedure of the Boltzmann equation in kinetic theory, the hydrodynamic limit of large particle systems in statistical mechanics, and early works on kinetic schemes to calculate shock waves and theoretical kinetic formulation for the pure hyperbolic case; see [16, 24, 26, 96, 107, 125, 126, 127, 144] and the references cited therein. In particular, a notion of kinetic solutions and a corresponding kinetic formulation have been extended. More precisely, consider the kinetic function, quasi-Maxwellian, χ on $\mathbb{R}^2$:

$$\chi(v;u) = \begin{cases} +1 & \text{for } \ 0 < v < u, \\ -1 & \text{for } \ u < v < 0, \\ \ 0 & \text{otherwise.} \end{cases} \tag{4.4}$$

If $u \in L^\infty\big(\mathbb{R}_+; L^1(\mathbb{R}^d)\big)$, then $\chi(v;u) \in L^\infty\big(\mathbb{R}_+; L^1(\mathbb{R}^{d+1})\big)$.

Definition. A function $u(t,\mathbf{x}) \in L^\infty\big(\mathbb{R}_+; L^1(\mathbb{R}^d)\big)$ is called a kinetic solution if $u(t,\mathbf{x})$ satisfies the following:

(i) The kinetic equation:

$$\partial_t \chi(v;u) + \mathbf{f}'(v)\cdot\nabla\chi(v;u) - \nabla\cdot(\mathbf{A}(v)\nabla\chi(v;u)) = \partial_v(m+n)(t,\mathbf{x},v) \tag{4.5}$$

holds in the sense of distributions with the initial data $\chi(v;u)|_{t=0} = \chi(v;u_0)$, for some nonnegative measures $m(t,\mathbf{x};v)$ and $n(t,\mathbf{x};v)$, where $n(t,\mathbf{x},v)$ is defined by

$$\langle\, n(t,\mathbf{x},\cdot), \psi(\cdot)\rangle := \sum_{k=1}^{d}\Big(\sum_{i=1}^{d}\partial_{x_i}\beta_{ik}^{\psi}(u(t,\mathbf{x}))\Big)^2 \in L^1(\mathbb{R}_+^{d+1}), \tag{4.6}$$

for any $\psi \in C_0^\infty(\mathbb{R})$ with $\psi \geq 0$ and $\beta_{ik}^{\psi}(u) := \int^u \sigma_{ik}(v)\sqrt{\psi(v)}\,dv$;

(ii) There exists $\mu \in L^\infty(\mathbb{R})$ with $0 \leq \mu(v) \to 0$ as $|v| \to \infty$ such that

$$\int_0^\infty\int_{\mathbb{R}^d}(m+n)(t,\mathbf{x};v)\,dt\,d\mathbf{x} \leq \mu(v); \tag{4.7}$$

(iii) For any two nonnegative functions $\psi_1,\ \psi_2 \in C_0^\infty(\mathbb{R})$,

$$\sqrt{\psi_1(u(t,\mathbf{x}))}\sum_{i=1}^{d}\partial_{x_i}\beta_{ik}^{\psi_2}(u(t,\mathbf{x})) = \sum_{i=1}^{d}\partial_{x_i}\beta_{ik}^{\psi_1\psi_2}(u(t,\mathbf{x})) \qquad a.e. \ (t,\mathbf{x}). \tag{4.8}$$

Then we have

- **Well-posedness in L^1**: Under this notion, the space L^1 is both a well-posed space for kinetic solutions and a well-defined space for the kinetic equation in (i). That is, the advantage of this notion is that the kinetic equation is well defined even when the macroscopic fluxes and diffusion matrices are not locally integrable so

that L^1 is a natural space on which the kinetic solutions are posed. This notion also covers the so-called renormalized solutions used in the context of scalar hyperbolic conservation laws by Bénilan-Carrillo-Wittbold [11]. Based on this notion, a new approach has been developed in [46] to establish the contraction property of kinetic solutions in L^1. This leads to a well-posedness theory—existence, stability, and uniqueness—for the Cauchy problem for kinetic solutions in L^1.

- **Consistency of the kinetic equation with the original macroscopic equation**: When the kinetic solution u is in L^∞, for any $\eta \in C^2$ with $\eta''(u) \geq 0$, multiplying $\eta'(v)$ both sides of the kinetic equation in (i) and then integrating in $v \in \mathbb{R}$ yield

$$\partial_t \eta(u) + \nabla_{\mathbf{x}} \cdot (\mathbf{q}(u) - \mathbf{A}(u)\nabla_{\mathbf{x}}\eta(u)) = -\int_{\mathbb{R}} \eta''(v)(m+n)(t,\mathbf{x};v)dv \leq 0.$$

  In particular, taking $\eta(u) = \pm u$ yields that u is a weak solution to the macroscopic equation. The uniqueness result implies that any kinetic solution in L^∞ must be an entropy solution. On the other hand, any entropy solution is actually a kinetic solution. Therefore, the two notions are equivalent for solutions in L^∞, although the notion of kinetic solutions is more general.

- For the isotropic case, condition (iii) automatically holds, which is actually a chain rule. In fact, the extension from the isotropic to anisotropic case is not a purely technical issue, and the fundamental and natural chain-rule type property (iii) does not appear in the isotropic case and turns out to be the corner-stone for the uniqueness in the anisotropic case. Moreover, condition (ii) implies that $m+n$ has no support at $u = \infty$.

Based on this notion, the corresponding kinetic formulation, and the uniqueness proof in the pure hyperbolic case in [126], we have developed a new effective approach to establish the contraction property of kinetic solutions in L^1. This leads to a well-posedness theory for the Cauchy problem of (3.13) with initial data:

$$u|_{t=0} = u_0 \in L^\infty(\mathbb{R}^d) \tag{4.9}$$

for kinetic solutions only in L^1.

Theorem 4.1 (Chen-Perthame [46])**.** (i) *For any kinetic solution* $u \in L^\infty(\mathbb{R}_+; L^1(\mathbb{R}^d))$ *with initial data* $u_0(\mathbf{x})$*, we have*

$$\|u(t,\cdot) - u_0\|_{L^1(\mathbb{R}^d)} \to 0 \qquad \text{as } t \to 0;$$

(ii) *If* $u, v \in L^\infty(\mathbb{R}_+; L^1(\mathbb{R}^d))$ *are kinetic solutions to* (4.1) *and* (4.9) *with initial data* $u_0(\mathbf{x})$ *and* $v_0(\mathbf{x})$*, respectively, then*

$$\|u(t,\cdot) - v(t,\cdot)\|_{L^1(\mathbb{R}^d)} \leq \|u_0 - v_0\|_{L^1(\mathbb{R}^d)}; \tag{4.10}$$

(iii) *For initial data* $u_0 \in L^1(\mathbb{R}^d)$*, there exists a unique kinetic solution* $u \in C\big(\mathbb{R}_+; L^1(\mathbb{R}^d)\big)$ *for the Cauchy problem* (3.13) *and* (4.9)*. If* $u_0 \in L^\infty \cap L^1(\mathbb{R}^d)$*, then the kinetic solution is the unique entropy solution and* $\|u(t,\cdot)\|_{L^\infty(\mathbb{R}^d)} \leq \|u_0\|_{L^\infty(\mathbb{R}^d)}$.

Furthermore, assume that the flux function $\mathbf{f}(u) \in C^1$ *and the diffusion matrix* $\mathbf{A}(u) \in C$ *satisfy the nonlinearity-diffusivity condition: The set*

$$\{v \, : \, \tau + \mathbf{f}'(v) \cdot \mathbf{y} = 0, \; \mathbf{y}\mathbf{A}(u)\,\mathbf{y}^\top = 0\} \subset \mathbb{R} \tag{4.11}$$

has zero Lebesgue measure, for any $\tau \in \mathbb{R}$, $\mathbf{y} = (y_1, \dots, y_d)$ *with* $|\mathbf{y}| = 1$. *Let* $u \in L^\infty(\mathbb{R}^{d+1}_+)$ *be the unique entropy solution to* (3.13) *and* (4.9) *with periodic initial data* $u_0 \in L^\infty$ *for period* $\mathbb{T}_P = \Pi_{i=1}^d [0, P_i]$, *i.e.,* $u_0(\mathbf{x}+P_i\mathbf{e}_i) = u_0(\mathbf{x})$ *a.e., where* $\{\mathbf{e}_i\}_{i=1}^d$ *is the standard basis in* $\mathbb{R}^d$. *Then*

$$\big\|u(t,\cdot) - \frac{1}{|\mathbb{T}_P|}\int_{\mathbb{T}_P} u_0(\mathbf{x})\, d\mathbf{x}\big\|_{L^1(\mathbb{T}_P)} \to 0 \qquad \text{as } \; t \to \infty, \tag{4.12}$$

where $|\mathbb{T}_P|$ *is the volume of the period* $\mathbb{T}_P$.

The *nonlinearity-diffusivity condition* implies that there is no interval of v in which the flux function $\mathbf{f}(v)$ is affine and the diffusion matrix $\mathbf{A}(v)$ is degenerate. Unlike the pure hyperbolic case, equation (4.1) is no longer self-similar invariant, and the diffusion term in the equation significantly affects the behavior of solutions; thus the argument in Chen-Frid [40] based on the self-similar scaling for the pure hyperbolic case could not be directly applied. The argument for proving Theorem 4.1 is based on the kinetic approach developed in [46], involves a time-scaling and a monotonicity-in-time of entropy solutions, and employs the advantages of the kinetic equation (4.5), in order to recognize the role of nonlinearity-diffusivity of equation (4.1) (see [47]).

Based on the very construction of the kinetic approach, the results can easily be translated in terms of the old Kruzkov entropies by integrating in v. In the case of uniqueness for the general case, this was performed in Bendahmane-Karlsen [10].

Follow-up results based on the Chen-Perthame approach in [46] include L^1-error estimates and continuous dependence of solutions in the convection function and the diffusion matrix in Chen-Karlsen [42]; and more general degenerate diffusion-advection-reaction equations in Chen-Karlsen [43]. Other recent developments include the related notion of dissipative solutions in Perthame-Souganidis [128], regularity results of solutions in Tadmor-Tao [146], as well as different types of diffusion terms in Andreianov-Bendahmane-Karlsen [5].

5. Nonlinear Equations of Mixed Hyperbolic-Elliptic Type

Unlike the linear case, very often, nonlinear equations can not be separated as two degenerate equations, but are truly mixed. Nonlinear partial differential equations of mixed hyperbolic-elliptic type arise naturally in many fundamental problems. In this section, we present two approaches through several examples to handle nonlinear mixed problems, especially nonlinear degenerate elliptic problems: Free-boundary techniques and weak convergence methods.

5.1. **Weak Convergence Methods.** We first present two problems: transonic flow pass obstacles and isometric embedding, for which weak convergence methods, especially methods of compensated compactness, play an important role.

5.1.1. *Transonic Flow Pass Obstacles in Gas Dynamics.* By scaling, the Euler equations for compressible, isentropic, irrotational fluids take the form:

$$\begin{cases} \partial_x(\rho u) + \partial_y(\rho v) = 0, \\ \partial_x v - \partial_y u = 0, \end{cases} \tag{5.1}$$

combined with the Bernoulli relation:

$$\rho = \Big(1 - \frac{\gamma - 1}{2}(u^2 + v^2)\Big)^{\frac{1}{\gamma-1}}, \qquad \gamma > 1, \tag{5.2}$$

for the pressure-density relation: $p = p(\rho) = \rho^\gamma/\gamma$. This provides two equations for the two unknowns (u, v). Furthermore, we note that, if ρ is constant (which is the incompressible case), the two equations in (5.1) become the Cauchy-Riemann equations for which any boundary value problem can be posed for the elliptic partial differential equations.

By the second equation in (5.1), we introduce the velocity potential φ:

$$(u, v) = \nabla\varphi. \tag{5.3}$$

Then our conservation law of mass becomes a nonlinear partial differential equation of second-order for φ:

$$\partial_x(\rho\,\partial_x\varphi) + \partial_y(\rho\,\partial_y\varphi) = 0, \tag{5.4}$$

which is combined with the Bernoulli relation:

$$\rho = \Big(1 - \gamma - \frac{1}{2}|\nabla\varphi|^2\Big)^{\frac{1}{\gamma-1}}. \tag{5.5}$$

Introduce the sound speed c as

$$c^2 = p'(\rho) = 1 - \frac{\gamma - 1}{2}q^2, \qquad \text{with the fluid speed } q = |\nabla\varphi| = \sqrt{u^2 + v^2}, \tag{5.6}$$

so that, at the sonic value when $q = c$, we have $q = q_*$ with

$$q_* := \sqrt{\frac{2}{\gamma + 1}}. \tag{5.7}$$

Then equation (5.4) is elliptic if

$$q < q_*$$

and hyperbolic when

$$q > q_*.$$

There is an upper bound placed on q from the Bernoulli relation:

$$q \le q_{cav} := \sqrt{\frac{2}{\gamma - 1}}, \tag{5.8}$$

where q_{cav} is the cavitation speed for which $\rho = 0$.

On the other hand, equation (5.4) corresponds to the Euler-Lagrange equation for the functional

$$\int_\Omega G(|\nabla\varphi|)dxdy, \tag{5.9}$$

where

$$G(q) = \int^{q^2} \Big(1 - \frac{\gamma - 1}{2}s\Big)^{\frac{1}{\gamma-1}} ds. \tag{5.10}$$

Since $(q_* - q)G''(q) > 0$, the direct method of calculus of variations (e.g. Evans [69]) provides the existence of weak solutions if it is known apriori that the flow is subsonic ($q < q_*$) so that G is convex and the problem is elliptic. For example, this includes the fundamental problem of subsonic flow around a profile as formulated in Bers [13].

A profile $\mathcal{P}$ is prescribed by a smooth curve, except for a trailing edge with an opening $\varepsilon\pi$ at z_T, $0 \le \varepsilon \le 1$.

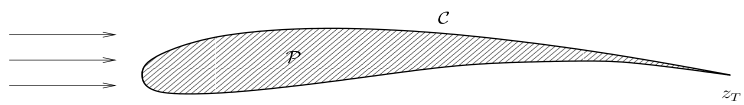

FIGURE 2. Profile $\mathcal{C}$ of the obstacle $\mathcal{P}$: $\varepsilon = 0$

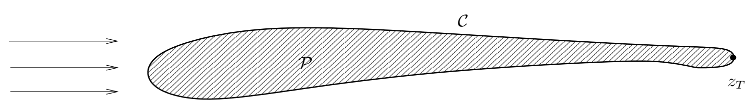

FIGURE 3. Profile $\mathcal{C}$ of the obstacle $\mathcal{P}$: $\varepsilon = 1$

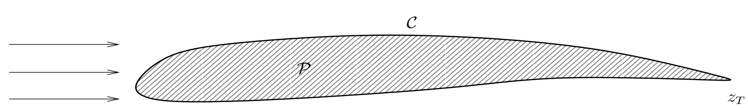

FIGURE 4. Profile $\mathcal{C}$ of the obstacle $\mathcal{P}$: $0 < \varepsilon < 1$

If $\varepsilon = 0$, the profile has a tangent at the trailing edge (see Fig. 2). The tangent to $\mathcal{P}$ satisfies a uniform Hölder condition with respect to the arc length. The velocity $\mathbf{w} = (u, v)$ must attain a given subsonic limit at infinity. We enforce the Kutta-Joukowski condition:

$$\begin{aligned} q &\to 0 && \text{as } (x, y) \to z_T \text{ if } \varepsilon = 1, \\ q &= O(1) && \text{as } (x, y) \to z_T \text{ if } 0 \le \varepsilon < 1, \end{aligned}$$

and define problem $P_1(\mathbf{w}_\infty)$ with a prescribed constant velocity $\mathbf{w}_\infty$ at infinity.

For a smooth profile, $\varepsilon = 1$ (see Fig. 3), define the circulation

$$\Gamma = \oint_{\mathcal{P}} (u, v) \cdot \boldsymbol{\tau}\, ds, \tag{5.11}$$

where $\boldsymbol{\tau}$ is the unit tangent to $\mathcal{P}$. In this case, we can also consider problem $P_2(\mathbf{w}_\infty, \Gamma)$ where the circulation is prescribed, instead of the Kutta-Joukowski condition.

In both problems, the slip boundary condition on the profile is required:

$$(u, v) \cdot \mathbf{n} = 0 \qquad \text{on } \mathcal{P} \qquad \text{(boundary condition)} \tag{5.12}$$

where $\mathbf{n}$ is the exterior unit normal on $\mathcal{P}$.

The first existence theorem for $P_1(\mathbf{w}_\infty)$ was given by Frankl-Keldysh [74] for sufficiently small speed at infinity. For a general gas, the first complete existence theorem for $P_2(\mathbf{w}_\infty, \Gamma)$ was given by Shiffman [141]. This was followed by a complete existence and uniqueness theorem by Bers [12] for $P_1(\mathbf{w}_\infty)$, a stronger uniqueness result of Finn-Gilbarg [73], and a higher dimensional result of Dong [64]. The basic result is as follows:

For a given constant velocity at infinity, there exists a number $\hat{q}$ depending upon the profile $\mathcal{P}$ and the adiabatic exponent $\gamma > 1$ such that the problem $P_1(\mathbf{w}_\infty)$ has a unique solution for $0 < q_\infty := |\mathbf{w}_\infty| < \hat{q}$. The maximum q_m of q takes on all values between 0 and q_*, $q_m \to 0$ as $q_\infty \to 0$, and $q_m \to q_*$ as $q_\infty \to \hat{q}$. A similar result holds for $P_2(\mathbf{w}_\infty, \Gamma)$.

The main tool for the results is to know apriori that, if $q_\infty < \hat{q}$ (i.e., the speed at infinity is not only subsonic) but sufficiently subsonic, then $q < q_*$ in the whole flow domain. Subsonic flow at infinity itself does not guarantee that the flow remains subsonic, since the profile produces flow orthogonal to the original flow direction. Shiffman's proof did use the direct method of the calculus of variations, while Bers's relied on both elliptic methods and the theory of pseudo-analytic functions. The existence of a critical point for the variational problem would be a natural goal for the case when q_∞ is not restricted to be less than $\hat{q}$, since it would provide a direct proof of our boundary value problem. However, no such proof has been given.

More recent investigations based on weak convergence methods start in the 1980's. DiPerna [63] suggested that the Murat-Tartar method of compensated compactness be amenable to flows which exhibit both elliptic and hyperbolic regimes, and investigated an asymptotic approximation to our system called the steady transonic small disturbance equation (TSD). He proved that, if a list of assumptions were satisfied (which then guaranteed the applicability of the method of compensated compactness), then a weak solution exists to the steady TSD equation. However, no one has ever been able to show that DiPerna's list is indeed satisfied.

In [115] (also see [117]), Morawetz layed out a program for proving the existence of the steady transonic flow problem about a bump profile in the upper half plane (which is equivalent to a symmetric profile in the whole plane). In particular, Morawetz showed that, if the key hypotheses of the method of compensated compactness could be satisfied, now known as a "compactness framework" (see Chen [28]), then indeed there would exist a weak solution to the problem of flow over a bump which is exhibited by subsonic and supersonic regimes, i.e., transonic flow.

The "compactness framework" for our system can be stated as follows: A sequence of functions $\mathbf{w}^\varepsilon(x,y) = (u^\varepsilon, v^\varepsilon)(x,y)$ defined on an open set $\Omega \subset \mathbb{R}^2$ satisfies the following set of conditions:

(A.1) $q^\varepsilon(x,y) = |\mathbf{w}^\varepsilon(x,y)| \le q_*$ a.e. in Ω for some positive constant $q_* < q_{cav}$;

(A.2) $\partial_x Q_{1\pm}(\mathbf{w}^\varepsilon) + \partial_y Q_{2\pm}(\mathbf{w}^\varepsilon)$ are confined in a compact set in $H^{-1}_{\mathrm{loc}}(\Omega)$ for entropy pairs $(Q_{1\pm}, Q_{2\pm})$, and $(Q_{1\pm}(\mathbf{w}^\varepsilon), Q_{2\pm}(\mathbf{w}^\varepsilon))$ are confined to a bounded set uniformly in $L^\infty_{\mathrm{loc}}(\Omega)$, where (Q_1, Q_2) is an entropy pair, that is, $\partial_x Q_1(\mathbf{w}^\varepsilon) + \partial_y Q_2(\mathbf{w}^\varepsilon) = 0$ along smooth solutions of our system).

In case (A.1) and (A.2) hold, then the Young measure $\nu_{x,y}$ determined by the uniformly bounded sequence of functions $\mathbf{w}^\varepsilon(x,y)$ is constrained by the following commutator relation:

$$\langle \nu_{x,y},\ Q_{1+}Q_{2-} - Q_{1-}Q_{2+}\rangle = \langle \nu_{x,y},\ Q_{1+}\rangle\langle \nu_{x,y},\ Q_{2-}\rangle - \langle \nu_{x,y},\ Q_{1-}\rangle\langle \nu_{x,y},\ Q_{2+}\rangle. \tag{5.13}$$

The main point for the compensated compactness framework is to prove that $\nu_{x,y}$ is a Dirac measure by using entropy pairs, which implies the compactness of the sequence $\mathbf{w}^\varepsilon(x,y) = (u^\varepsilon, v^\varepsilon)(x,y)$ in $L^1_{\rm loc}(\Omega)$. In this context, both DiPerna [63] and Morawetz [115] needed to presume the existence of an approximating sequence parameterized by ε to their problems satisfying (A.1) and (A.2) so that they could exploit the commutator relation and obtain the strong convergence in $L^1_{\rm loc}(\Omega)$ to a weak solution of their problems.

As it turns out, there is one problem where (A.1) and (A.2) hold trivially, i.e., the sonic limit of subsonic flows. In that case, we return to the result by Bers [12] and Shiffman [141], which says that, if the speed at infinity q_∞ is less than some $\hat{q}$, there is a smooth unique solution to problems $P_1(\mathbf{w}_\infty)$ and $P_2(\mathbf{w}_\infty, \Gamma)$ and ask what happens as $q_\infty \nearrow \hat{q}$. In this case, the flow develops sonic points and the governing equations become degenerate elliptic. Thus, if we set $\varepsilon = \hat{q} - q_\infty$ and examine a sequence of exact smooth solutions to our system, we see trivially that (A.1) is satisfied since $|q^\varepsilon| \le q_*$, and (A.2) is satisfied since $\partial_x Q_\pm(\mathbf{w}^\varepsilon) + \partial_y Q_\pm(\mathbf{w}^\varepsilon) = 0$ along our solution sequence. The effort is in finding entropy pairs which can guarantee that the Young measure $\nu_{x,y}$ reduces to a Dirac mass. Ironically, the original conservation equations of momentum in fact provide two sets of entropy pairs, while the irrotationality and mass conservation equations provide another two sets. This observation has been explored in detail in Chen-Dafermos-Slemrod-Wang [34].

What then about the fully transonic problem of flow past an obstacle or bump where $q_\infty > \hat{q}$? In Chen-Slemrod-Wang [48], we have provided some of the ingredients for satisfying (A.1) and (A.2). More precisely, we have introduced the usual flow angle $\theta = \tan^{-1}(\frac{v}{u})$ and written the irrotationality and mass conservation equation as an artificially viscous problem:

$$\begin{cases} \partial_x v^\varepsilon - \partial_y u^\varepsilon = \varepsilon \Delta\theta^\varepsilon, \\ \partial_x(\rho^\varepsilon u^\varepsilon) + \partial_y(\rho^\varepsilon v^\varepsilon) = \varepsilon \nabla\cdot(\sigma(\rho^\varepsilon)\nabla\rho^\varepsilon), \end{cases} \tag{5.14}$$

where $\sigma(\rho)$ is suitably chosen, and appropriate boundary conditions are imposed for this regularized "viscous" problem. The crucial point is that a uniformly L^∞ bound in q^ε has been obtained when $1 \le \gamma < 3$ which uniformly prevents cavitation. However, in this formulation, a uniform bound in the flow angle θ^ε and lower bound of q^ε away from zero (stagnation) in any fixed region disjoint from the profile are still assumed apriori to guarantee the (q,θ)–version of (A.1). On the other hand, (A.2) is easily obtained from the viscous formulation by using a special entropy pair of Osher-Hafez-Whitlow [124]. In fact, this entropy pair is very important: It guarantees that the inviscid limit of the above viscous system satisfies a physically meaningful "entropy" condition (Theorem 2 of [124]). With (A.1) and (A.2) satisfied, Morawetz's theory [115] then applies to yield the strong convergence in $L^1_{\rm loc}(\Omega)$ of our approximating sequence. It would be interesting to establish a uniform bound in the flow angle θ^ε and lower bound of q^ε away from zero (stagnation) in any fixed region disjoint from the profile under some natural conditions on the profile.

5.1.2. *Isometric Embeddings in Differential Geometry.* In differential geometry, a long-standing, fundamental problem is to characterize intrinsic metrics on a two-dimensional surface $\mathcal{M}^2$ which can be realized as embeddings into $\mathbb{R}^3$. Let $\emptyset \subset \mathbb{R}^2$ be an open set and $g_{ij}, i, j = 1, 2$, be a given matric on $\mathcal{M}^2$ parameterized on Ω. Then the first fundamental form for $\mathcal{M}^2$ is

$$I = g_{11}(dx)^2 + 2g_{12}dxdy + g_{22}(dy)^2. \tag{5.15}$$

Isometric Embedding Problem. *Seek an injective map* $\mathbf{r} : \Omega \to \mathbb{R}^3$ *such that* $d\mathbf{r} \cdot d\mathbf{r} = I$*, i.e.,*

$$\partial_x \mathbf{r} \cdot \partial_x \mathbf{r} = g_{11}, \quad \partial_x \mathbf{r} \cdot \partial_y \mathbf{r} = g_{12}, \quad \partial_y \mathbf{r} \cdot \partial_y \mathbf{r} = g_{22}, \tag{5.16}$$

so that $(\partial_x \mathbf{r}, \partial_y \mathbf{r})$ *in* $\mathbb{R}^3$ *are linearly independent.*

The equations above are three nonlinear partial differential equations for the three components of $\mathbf{r}$. Recall that the second fundamental form II for $\mathcal{M}^2$ defined on Ω is

$$II = -d\mathbf{n} \cdot d\mathbf{r} = h_{11}(dx)^2 + 2h_{12}dxdy + h_{22}(dy)^2, \tag{5.17}$$

and h_{ij} is the orthogonality of the unit normal $\mathbf{n}$ of the surface $\mathbf{r}(\Omega) \subset \mathbb{R}^3$ to its tangent plane. The Christoffel symbols are

$$\Gamma_{ij}^{(k)} := \frac{1}{2} g^{kl} \left(\partial_j g_{il} + \partial_i g_{jl} - \partial_l g_{ij} \right),$$

which depend on the first derivatives of g_{ij}, where the summation convention is used, (g^{kl}) denotes the inverse of (g_{ij}), and $(\partial_1, \partial_2) := (\partial_x, \partial_y)$.

The fundamental theorem of surface theory indicates that there exists a surface in $\mathbb{R}^3$ whose first and second fundamental forms are I and II if the coefficients (g_{ij}) and (h_{ij}) of the two given quadratic forms I and II, I being positive definite, satisfy the Gauss-Codazzi system. This theorem also holds even for discontinuous coefficients h_{ij} (cf. Mardare [114]).

Given (g_{ij}), the coefficients (h_{ij}) of II are determined by the *Gauss-Codazzi system*, which consists of the Codazzi equations:

$$\begin{cases} \partial_x M - \partial_y L = \Gamma_{22}^{(2)} L - 2\Gamma_{12}^{(2)} M + \Gamma_{11}^{(2)} N, \\ \partial_x N - \partial_y M = -\Gamma_{22}^{(1)} L + 2\Gamma_{12}^{(1)} M - \Gamma_{11}^{(1)} N, \end{cases} \tag{5.18}$$

and the Gauss equation, i.e., *Monge-Ampère* constraint:

$$LN - M^2 = K, \tag{5.19}$$

where

$$L = \frac{h_{11}}{\sqrt{|g|}}, \quad M = \frac{h_{12}}{\sqrt{|g|}}, \quad N = \frac{h_{22}}{\sqrt{|g|}}, \quad |g| = det(g_{ij}) = g_{11}g_{22} - g_{12}^2, \tag{5.20}$$

$K(x, y)$ is the Gauss curvature that is determined by the relation:

$$K(x, y) = \frac{R_{1212}}{|g|}, \tag{5.21}$$

and

$$R_{ijkl} = g_{lm} \big(\partial_k \Gamma_{ij}^{(m)} - \partial_j \Gamma_{ik}^{(m)} + \Gamma_{ij}^{(n)} \Gamma_{nk}^{(m)} - \Gamma_{ik}^{(n)} \Gamma_{nj}^{(m)} \big)$$

is the Riemann curvature tensor depending on g_{ij} and its first and second derivatives.

Therefore, given a positive definite (g_{ij}), the Gauss-Codazzi system consists of the three equations for the three unknowns (L, M, N) determining the second fundamental form. Note that, while (g_{ij}) is positive definite, R_{1212} may have any sign and hence K may have any sign.

From the viewpoint of geometry, the Gauss equation is a Monge-Ampère equation and the Codazzi equations are as integrability relations. On the other hand, we are interested in a fluid mechanic formulation for the isometric embedding problem so that the problem may be solved via the approaches that have shown to be useful in fluid mechanics, especially for the nonlinear partial differential equations of mixed hyperbolic-elliptic type. To achieve this, the way to reformulate the problem via solvability of the Codazzi equations under the Gauss equation has been adopted in Chen-Slemrod-Wang [49].

Set

$$L = \rho v^2 + p, \quad M = -\rho uv, \quad N = \rho u^2 + p,$$

choose the "pressure" p as for the Chaplygin type gas: $p = -\frac{1}{\rho}$, and set $q^2 = u^2 + v^2$. Then the Codazzi equations become the balance equations of momentum:

$$\begin{aligned} \partial_x(\rho uv) + \partial_y(\rho v^2 + p) &= -\Gamma_{22}^{(2)}(\rho v^2 + p) - 2\Gamma_{12}^{(2)}\rho uv - \Gamma_{11}^{(2)}(\rho u^2 + p), \\ \partial_x(\rho u^2 + p) + \partial_y(\rho uv) &= -\Gamma_{22}^{(1)}(\rho v^2 + p) - 2\Gamma_{12}^{(1)}\rho uv - \Gamma_{11}^{(1)}(\rho u^2 + p), \end{aligned} \tag{5.22}$$

and the Gauss equation becomes the Bernoulli law:

$$\rho = \frac{1}{\sqrt{q^2 + K}}. \tag{5.23}$$

Set the "sound" speed: $c = \sqrt{p'(\rho)}$, i.e., $c^2 = \frac{1}{\rho^2} = q^2 + K$. Then

- When $K > 0$, $q^2 < c^2$ and the "flow" is subsonic;
- When $K < 0$, $q^2 > c^2$ and the "flow" is supersonic;
- When $K = 0$, $q^2 = c^2$ and the "flow" is sonic.

K<0

K>0

FIGURE 5. The Gaussian curvature on a torus: Doughnut surface or toroidal shell

Therefore, the existence of an isometric immersion is equivalent to the existence of a weak solution of the balance equations of momentum in the fluid mechanic formulation,

which are nonlinear partial differential equations of mixed hyperbolic-elliptic type. Many usual surfaces have their Gauss curvature of changing sign, such as tori: Doughnut surfaces or toroidal shells in Fig. 5.

An appropriate approximate method to construct approximate solutions to the Gauss-Codazzi system has been designed, and a compensated compactness approach to establish the existence of a weak solution has been developed in [49]. The advantage of the compensated compactness approach is that it works for both the elliptic and hyperbolic phase. For more details about isometric embedding of Riemann manifolds in Euclidean spaces, we refer the reader to Han-Hong [83] and Yau [156].

For isometric embedding of a higher dimensional Riemannian manifold into the Euclidean space $\mathbb{R}^N$ with optimal dimension N (i.e., the Cartan-Janet dimension), the Gauss-Codazzi equations should be supplemented by the Ricci equations to describe the connection form on the normal bundle. The Gauss-Codazzi-Ricci system even has no type in general, although it inherits important geometric features, including the beautiful Div-Curl structure which yields its weak continuity (see [50]). Also see Bryant-Griffiths-Yang [18].

5.2. **Free-Boundary Techniques.** To explain the techniques, we focus on the shock reflection-diffraction problem for potential flow which is widely used in aerodynamics.

When a plane shock in the $(t, \mathbf{x})$-coordinates, $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$, with the left state $(\rho, \nabla_{\mathbf{x}}\Phi) = (\rho_1, u_1, 0)$ and the right state $(\rho_0, 0, 0), u_1 > 0, \rho_0 < \rho_1$, hits a symmetric wedge

$$W := \{\mathbf{x} : |x_2| < x_1 \tan\theta_w, x_1 > 0\}$$

head on, it experiences a reflection-diffraction process, where ρ is the density and Φ is the velocity potential of the fluid; see Fig. 1. Then a self-similar reflected shock moves outward as the original shock moves forward in time. The complexity of reflection-diffraction configurations was first reported by Ernst Mach in 1878, and experimental, computational, and asymptotic analysis has shown that various different patterns may occur, including regular and Mach reflection (cf. [9, 81, 103, 116, 137, 153]). However, most of the fundamental issues for shock reflection-diffraction have not been understood, including the global structure, stability, and transition of different patterns of shock reflection-diffraction configurations. Therefore, it becomes essential to establish the global existence and structural stability of solutions in order to understand fully the shock reflection-diffraction phenomena. Furthermore, this problem is also fundamental in the mathematical theory of multidimensional conservation laws since their solutions are building blocks and asymptotic attractors of general solutions to the two-dimensional Euler equations for compressible fluid flow (cf. Courant-Friedrichs [53], von Neumann [153], and Glimm-Majda [81]; also see [9, 28, 54, 103, 116, 137]). As we will show below, the problem involves nonlinear partial differential equations of mixed elliptic-hyperbolic type, along with the other challenging difficulties such as free boundary problems and corner singularities, especially when a free boundary meets an elliptic degenerate curve.

By scaling, the Euler equations for potential flow consist of the conservation law of mass and the Bernoulli law with the form:

$$\begin{cases} \partial_t\rho + \nabla\cdot(\rho\nabla\Phi) = 0, \\ \partial_t\Phi + \frac{1}{2}|\nabla\Phi|^2 + \frac{\rho^{\gamma-1}}{\gamma-1} = \frac{\rho_0^{\gamma-1}}{\gamma-1}, \end{cases} \tag{5.24}$$

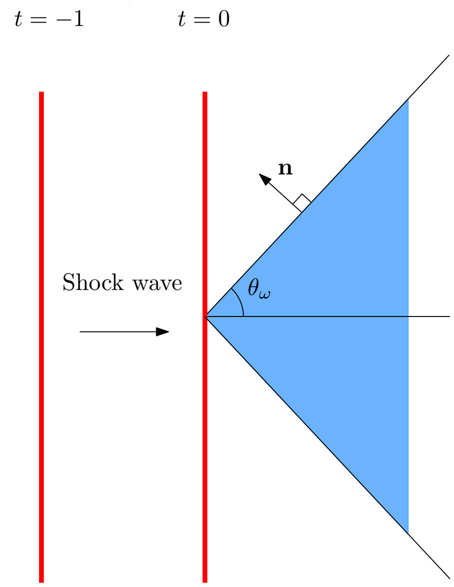

FIGURE 6. Shock reflection-diffraction problem

where $\gamma > 1$ is the adiabatic exponent of the fluid. Then the reflection-diffraction problem can be formulated as the following mathematical problem.

Initial-Boundary Value Problem. *Seek a solution of the system with the initial condition at $t = 0$:*

$$(\rho, \Phi)|_{t=0} = \begin{cases} (\rho_0, 0) & \textit{for } |x_2| > x_1 \tan\theta_w,\ x_1 > 0, \\ (\rho_1, u_1 x_1) & \textit{for } x_1 < 0, \end{cases} \tag{5.25}$$

and the slip boundary condition along the wedge boundary ∂W:

$$\nabla\Phi \cdot \mathbf{n}|_{\partial W} = 0, \tag{5.26}$$

where $\mathbf{n}$ is the exterior unit normal to ∂W.

Since the initial-boundary value problem is invariant under the self-similar scaling:

$$(t, \mathbf{x}) \to (\alpha t, \alpha \mathbf{x}), \quad (\rho, \Phi) \to (\rho, \frac{\Phi}{\alpha}) \qquad \text{for } \alpha \neq 0,$$

we seek self-similar solutions with the form:

$$\rho(t, \mathbf{x}) = \rho(\xi, \eta), \quad \Phi(t, \mathbf{x}) = t\,\psi(\xi, \eta) \qquad \text{for} \quad (\xi, \eta) = \frac{\mathbf{x}}{t}.$$

Then the pseudo-potential function $\varphi = \psi - \frac{1}{2}(\xi^2 + \eta^2)$ is governed by the following potential flow equation of second-order:

$$\operatorname{div}\big(\rho(|D\varphi|^2, \varphi) D\varphi\big) + 2\rho(|D\varphi|^2, \varphi) = 0 \tag{5.27}$$

with $\rho(|D\varphi|^2, \varphi) := \big(\rho_0^{\gamma-1} - (\gamma+1)(\varphi + \frac{1}{2}|D\varphi|^2)\big)^{\frac{1}{\gamma-1}}$, where $D = (\partial_\xi, \partial_\eta)$. The sonic speed is

$$c = c(|D\varphi|^2, \varphi, \rho_0^{\gamma-1}) := \big(\rho_0^{\gamma-1} - (\gamma-1)(\varphi + \frac{1}{2}|D\varphi|^2)\big)^{1/2}. \tag{5.28}$$

Equation (5.27) is a *second-order nonlinear partial differential equations of mixed elliptic-hyperbolic type.* It is strictly *elliptic* (i.e., pseudo-subsonic) if

$$|D\varphi| < c(|D\varphi|^2, \varphi, \rho_0^{\gamma-1}), \tag{5.29}$$

and strictly *hyperbolic* (i.e., pseudo-supersonic) if $|D\varphi| > c(|D\varphi|^2, \varphi, \rho_0^{\gamma-1})$.

Shocks are discontinuities in the pseudo-velocity $D\varphi$. That is, if Ω^+ and $\Omega^- := \Omega \setminus \overline{\Omega^+}$ are two nonempty open subsets of $\Omega \subset \mathbb{R}^2$ and $S := \partial\Omega^+ \cap \Omega$ is a C^1 curve where $D\varphi$ has a jump, then $\varphi \in W^{1,1}_{loc}(\Omega) \cap C^1(\Omega^\pm \cup S) \cap C^2(\Omega^\pm)$ is a global weak solution of (5.27) in Ω in the sense of distributions if and only if φ is in $W^{1,\infty}_{loc}(\Omega)$ and satisfies equation (5.27) in $\Omega^\pm$ and the Rankine-Hugoniot condition on S:

$$\rho(|D\varphi|^2, \varphi) D\varphi \cdot \mathbf{n}\Big|_{S+} = \rho(|D\varphi|^2, \varphi) D\varphi \cdot \mathbf{n}\Big|_{S-}. \tag{5.30}$$

The continuity of φ is followed by the continuity of the tangential derivative of φ across S. The discontinuity S of $D\varphi$ is called a shock if φ further satisfies the physical entropy condition that the corresponding density function $\rho(|D\varphi|^2, \varphi)$ increases across S in the pseudo-flow direction.

The plane incident shock solution in the $(t, \mathbf{x})$–coordinates corresponds to a continuous weak solution φ in the self-similar coordinates (ξ, η) with the following form:

$$\varphi_0(\xi, \eta) = -\frac{1}{2}(\xi^2 + \eta^2) \qquad \text{for} \quad \xi > \xi_0, \tag{5.31}$$

$$\varphi_1(\xi, \eta) = -\frac{1}{2}(\xi^2 + \eta^2) + u_1(\xi - \xi_0) \qquad \text{for} \quad \xi < \xi_0, \tag{5.32}$$

respectively, where $\xi_0 > 0$ is the location of the incident shock, uniquely determined by (ρ_0, ρ_1, γ). Since the problem is symmetric with respect to the axis $\eta = 0$, it suffices to consider the problem in the half-plane $\eta > 0$ outside the half-wedge $\Lambda := \{\xi \le 0, \eta > 0\} \cup \{\eta > \xi \tan\theta_w,\, \xi > 0\}$. Then the initial-boundary value problem can be formulated as the boundary value problem in Λ in the coordinates (ξ, η).

Boundary Value Problem. *Seek a solution φ of equation* (5.27) *in the unbounded domain Λ with the slip boundary condition on the wedge boundary $\partial\Lambda$:*

$$D\varphi \cdot \mathbf{n}|_{\partial\Lambda} = 0 \tag{5.33}$$

and the asymptotic boundary condition at infinity:

$$\varphi \to \hat{\varphi} := \begin{cases} \varphi_0 & \textit{for } \xi > \xi_0, \eta > \xi \tan\theta_w, \\ \varphi_1 & \textit{for } \xi < \xi_0, \eta > 0, \end{cases} \qquad \textit{when } \xi^2 + \eta^2 \to \infty, \tag{5.34}$$

in the sense that $\lim_{R\to\infty} \|\varphi - \hat{\varphi}\|_{C(\Lambda \setminus B_R(0))} = 0$.

Since φ_1 does not satisfy the slip boundary condition, the solution must differ from φ_1 in $\{\xi < \xi_0\} \cap \Lambda$, and thus a shock diffraction by the wedge occurs. A local existence theory of regular shock reflection-diffraction near the reflection point P_0 can be established by following the von Neumann detachment criterion [153] with the structure of solution as in Fig. 7, when the wedge angle is large and close to $\frac{\pi}{2}$, in which the vertical line is the incident shock $S = \{\xi = \xi_0\}$ that hits the wedge at the point $P_0 = (\xi_0, \xi_0 \tan\theta_w)$, and state (0) and state (1) ahead of and behind S are given by φ_0 and φ_1, respectively. The solutions φ and φ_1 differ only in the domain $P_0P_1P_2P_3$ because of shock reflection

by the wedge boundary at P_0 and diffraction by the wedge vertex P_3, where the curve $P_0P_1P_2$ is the reflected shock with the straight segment P_0P_1. State (2) behind P_0P_1 can be computed explicitly with the form:

$$\varphi_2(\xi,\eta) = -\frac{1}{2}(\xi^2+\eta^2) + u_2(\xi-\xi_0) + (\eta-\xi_0\tan\theta_w)u_2\tan\theta_w, \tag{5.35}$$

which satisfies $D\varphi\cdot\mathbf{n}=0$ on $\partial\Lambda\cap\{\xi>0\}$; the constant velocity u_2 and the slope of P_0P_1 are determined by $(\theta_w,\rho_0,\rho_1,\gamma)$ from the two algebraic equations expressing (5.30) and continuous matching of states (1) and (2) across P_0P_1, whose existence is exactly guaranteed by the condition on $(\theta_w,\rho_0,\rho_1,\gamma)$ which is necessary for the existence of a global regular shock reflection-diffraction configuration.

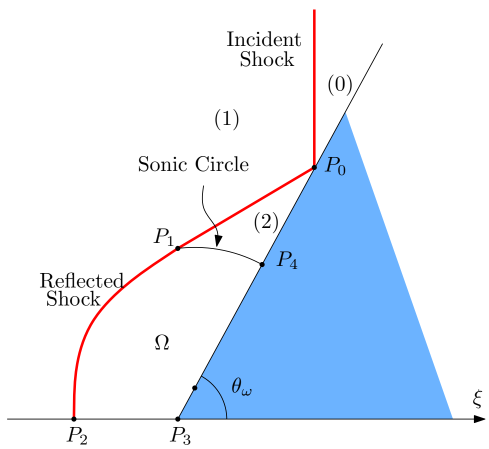

FIGURE 7. Regular reflection-diffraction configuration

A rigorous mathematical approach has been developed in Chen-Feldman [38] to extend the local theory to a global theory for solutions of regular shock reflection-diffraction, which converge to the unique solution of the normal shock reflection when θ_w tends to $\frac{\pi}{2}$. The solution φ is pseudo-subsonic within the sonic circle for state (2) with center $(u_2, u_2\tan\theta_w)$ and radius $c_2>0$ (the sonic speed) and is pseudo-supersonic outside this circle containing the arc P_1P_4 in Fig. 7, so that φ_2 is the unique solution in the domain $P_0P_1P_4$. In the domain Ω, the solution is pseudo-subsonic, smooth, and $C^{1,1}$-matching with state (2) across P_1P_4 and satisfies $\varphi_\eta=0$ on P_2P_3; the transonic shock curve P_1P_2 matches up to second-order with P_0P_1 continuously and is orthogonal to the ξ-axis at the point P_2 so that the standard reflection about the ξ–axis yields a global solution in the whole plane.

One of the main difficulties for the global existence is that the ellipticity condition for (5.27) is hard to control. The second difficulty is that the ellipticity degenerates at the sonic circle P_1P_4. The third difficulty is that, on P_1P_4, the solution in Ω has to be matched with φ_2 at least in C^1, i.e., the two conditions on the fixed boundary P_1P_4: the Dirichlet and conormal conditions, which are generically overdetermined for an elliptic equation since the conditions on the other parts of boundary are prescribed. Thus it is required to prove that, if φ satisfies (5.27) in Ω, the Dirichlet continuity condition on the sonic circle, and the appropriate conditions on the other parts of $\partial\Omega$ derived from the boundary

value problem, then the normal derivative $D\varphi \cdot \mathbf{n}$ automatically matches with $D\varphi_2 \cdot \mathbf{n}$ along P_1P_4. Indeed, equation (5.27), written in terms of the function $w = \varphi - \varphi_2$ in the (x, y)–coordinates defined near P_1P_4 such that P_1P_4 becomes a segment on $\{x = 0\}$, has the form:

$$\big(2x - (\gamma+1)\partial_x w\big)\partial_{xx} w + \frac{1}{c_2^2}\partial_{yy} w - \partial_x w = 0 \qquad \text{in } x > 0 \text{ and near } x = 0, \tag{5.36}$$

plus the "small" terms that are controlled by $\frac{\pi}{2} - \theta_w$ in appropriate norms. Equation (5.36) is *strictly elliptic* if $\partial_x w < \frac{2x}{\gamma+1}$ and become *degenerate elliptic* when $\partial_x w = \frac{2x}{\gamma+1}$. Hence, it is required to obtain the $C^{1,1}$–estimates near P_1P_4 to ensure $|\partial_x w| < \frac{2x}{\gamma+1}$ which in turn implies both the ellipticity of the equation in Ω and the match of normal derivatives $D\varphi \cdot \mathbf{n} = D\varphi_2 \cdot \mathbf{n}$ along P_1P_4. Taking into account the "small" terms to be added to equation (5.36), it is needed to make the stronger estimate $|\partial_x w| \leq \frac{4x}{3(\gamma+1)}$ and assume that $\frac{\pi}{2} - \theta_w$ is suitably small to control these additional terms. Another issue is the non-variational structure and nonlinearity of this problem which makes it hard to apply directly the approaches of Caffarelli [20] and Alt-Caffarelli-Friedman [3, 4]. Moreover, the elliptic degeneracy and geometry of the problem makes it difficult to apply the hodograph transform approach in Kinderlehrer-Nirenberg [95] and Chen-Feldman [37] to fix the free boundary.

For these reasons, one of the new ingredients in the approach is to develop further the iteration scheme in [36] to a partially modified equation. Equation (5.36) is modified in Ω by a proper Shiffmanization (i.e. cutoff) that depends on the distance to the sonic circle, so that the original and modified equations coincide for φ satisfying $|\partial_x w| \leq \frac{4x}{3(\gamma+1)}$, and the modified equation $\mathcal{N}\varphi = 0$ is elliptic in Ω with elliptic degeneracy on P_1P_4. Then a free boundary problem is solved for this modified equation: The free boundary is the curve P_1P_2, and the free boundary conditions on P_1P_2 are $\varphi = \varphi_1$ and the Rankine-Hugoniot condition (5.30). Moreover, the precise gradient estimate: $|\partial_x w| < \frac{4x}{3(\gamma+1)}$ is made to ensure that φ satisfies the original equation (5.27).

This global theory for large-angle wedges has been extended in Chen-Feldman [39] up to the sonic angle $\theta_s \leq \theta_c$, i.e. state (2) is sonic when $\theta_w = \theta_s$, such that, as long as $\theta_w \in (\theta_s, \frac{\pi}{2}]$, the global regular shock reflection-diffraction configuration exists, which solves the von Neumann's sonic conjecture (1943) [153] when $u_1 < c_1$. Furthermore, the optimal regularity of regular reflection-diffraction solutions near the pseudo-sonic circle has been shown in Bae-Chen-Feldman [6] to be $C^{1,1}$ as established in [38, 39].

Some important efforts were made mathematically for the global reflection problem via simplified models, including the unsteady TSD equation and other nonlinear models (cf. [94]). Furthermore, in order to deal with the reflection problem, some asymptotic methods have been also developed including the work by Lighthill, Keller, Morawetz, and others. The physicality of weak Prandtl-Meyer reflection for supersonic potential flow around a ramp has been also analyzed in [66]. Another recent effort has been made on various important physical, mixed elliptic-hyperbolic problems in steady potential flow for which great progress has been made (cf. [28, 33, 39, 117] and the references cited therein).

The self-similar solutions for the full Euler equations are required for the Mach reflection-diffraction configurations and general two-dimensional Riemann problems. For this case, the self-similar solutions are governed by a system that consists of two transport equations

and two *nonlinear equations of mixed hyperbolic-elliptic type*. One of the important features in the reflection-diffraction configurations is that the Euler equations for potential flow and the full Euler equations coincide in some important regions of the solutions.

6. Singular Limits to Nonlinear Degenerate Hyperbolic Equations

One of the important singular limit problems is the vanishing viscosity limit problem for the Navier-Stokes equations to the Euler equations for compressible barotropic fluids. The Navier-Stokes equations for a compressible viscous, barotropic fluid in Eulerian coordinates in $\mathbb{R}^2_+$ take the following form:

$$\begin{cases} \partial_t\rho + \partial_x(\rho v) = 0, \\ \partial_t(\rho v) + \partial_x(\rho v^2 + p) = \varepsilon \partial_{xx} v, \end{cases} \tag{6.1}$$

with the initial conditions:

$$\rho(0,x) = \rho_0(x), \qquad v(0,x) = v_0(x) \tag{6.2}$$

such that $\lim_{x\to\pm\infty}(\rho_0(x), v_0(x)) = (\rho^\pm, v^\pm)$, where ρ denotes the density, v represents the velocity of the fluid when $\rho > 0$, p is the pressure, $m = \rho v$ is the momentum, and $(\rho^\pm, v^\pm)$ are constant states with $\rho^\pm > 0$. The physical viscosity coefficient ε is restricted to $\varepsilon \in (0, \varepsilon_0]$ for some fixed $\varepsilon_0 > 0$. For a polytropic perfect gases, the pressure-density relation is determined by (3.14)–(3.15) with adiabatic exponent $\gamma > 1$.

Formally, when $\varepsilon \to 0$, the Navier-Stokes equations become the isentropic Euler equations (3.13), which is strictly hyperbolic when $\rho > 0$. However, near the vacuum $\rho = 0$, the two characteristic speeds of (3.13) may coincide and the system be degenerate hyperbolic.

The vanishing artificial/numerical viscosity limit to the isentropic Euler equations with general L^∞ initial data has been studied by DiPerna [61], Chen [29, 31], Ding [58], Ding-Chen-Luo [59], Lions-Perthame-Souganidis [106], and Lions-Perthame-Tadmor [107] via the methods of compensated compactness. Also see DiPerna [62], Morawetz [118], Perthame-Tzavaras [129], and Serre [138] for the vanishing artificial/numerical viscosity limit to general 2×2 strictly hyperbolic systems of conservation laws. The vanishing artificial viscosity limit to general strictly hyperbolic systems of conservation laws with general small BV initial data was first established by Bianchini-Bressan [15] via direct BV estimates with small oscillation. Also see LeFloch-Westdickenberg [104] for the existence of finite-energy solutions to the isentropic Euler equations with finite-energy initial data for the case $1 < \gamma \le 5/3$.

The idea of regarding inviscid gases as viscous gases with vanishing real physical viscosity can date back the seminal paper by Stokes [145] and the important contribution of Rankine [132], Hugoniot [85], Rayleigh [133], and Taylor [149] (cf. Dafermos [54]). However, the first rigorous convergence analysis of vanishing physical viscosity from the Navier-Stokes equations (6.1) to the isentropic Euler equations was made by Gilbarg [77] in 1951, when he established the mathematical existence and vanishing viscous limit of the Navier-Stokes shock layers. For the convergence analysis confined in the framework of piecewise smooth solutions; see Hoff-Liu [87], Gùes-Métivier-Williams-Zumbrun [82], and the references cited therein. The convergence of vanishing physical viscosity with general initial data was first studied by Serre-Shearer [139] for a 2×2 system in nonlinear elasticity with severe growth conditions on the nonlinear function in the system.

In Chen-Perepelitsa [45], we have first developed new uniform estimates with respect to the real physical viscosity coefficient for the solutions of the Navier-Stokes equations with the finite-energy initial data and established the H^{-1}-compactness of weak entropy dissipation measures of the solutions of the Navier-Stokes equations for any weak entropy pairs generated by compactly supported C^2 test functions. With these, the existence of measure-valued solutions with possibly unbounded support has been established, which are confined by the Tartar-Murat commutator relation with respect to two pairs of weak entropy kernels. Then we have established the reduction of measure-valued solutions with unbounded support for the case $\gamma \geq 3$ and, as corollary, we have obtained the existence of global finite-energy entropy solutions of the Euler equations with general initial data for $\gamma \geq 3$. We have further simplified the reduction proof of measure-valued solutions with unbounded support for the case $1 < \gamma \leq 5/3$ in LeFloch-Westdickenberg [104] and extended to the whole interval $1 < \gamma < 3$. With all of these, we have established the first convergence result for the vanishing physical viscosity limit of solutions of the Navier-Stokes equations to a finite-energy entropy solution of the isentropic Euler equations with general finite-energy initial data. We remark that the existence of finite-energy solutions to the isentropic Euler equations for the case $\gamma > 5/3$ have been established, which is in addition to the result in [104] for $1 < \gamma \leq 5/3$.

More precisely, consider the Cauchy problem (6.1)–(6.2) for the Navier-Stokes equations in $\mathbb{R}^2_+$. Hoff's theorem in [86] (also see Kanel [90] for the case of the same end-states) indicates that, when the initial functions $(\rho_0(x), v_0(x))$ are smooth with the lower bounded density $\rho_0(x) \geq c_0^\varepsilon > 0$ for $x \in \mathbb{R}$ and

$$\lim_{x\to\pm\infty}(\rho_0(x), v_0(x)) = (\rho^\pm, v^\pm),$$

then there exists a unique smooth solution $(\rho^\varepsilon(t,x), v^\varepsilon(t,x))$, globally in time, with $\rho^\varepsilon(t,x) \geq c_\varepsilon(t)$ for some $c_\varepsilon(t) > 0$ for $t \geq 0$ and $\lim_{x\to\pm\infty}(\rho^\varepsilon(t,x), v^\varepsilon(t,x)) = (\rho^\pm, v^\pm)$.

Let $(\bar\rho(x), \bar v(x))$ be a pair of smooth monotone functions satisfying $(\bar\rho(x), \bar v(x)) = (\rho^\pm, v^\pm)$ when $\pm x \geq L_0$ for some large $L_0 > 0$. The total mechanical energy for (6.1) in $\mathbb{R}$ with respect to the pair $(\bar\rho, \bar v)$ is

$$E[\rho, v](t) := \int_{\mathbb{R}} \Big(\eta^*(\rho, m) - \eta^*(\bar\rho, \bar m) - \nabla\eta^*(\bar\rho, \bar m)\cdot(\rho - \bar\rho, m - \bar m)\Big)\, dx \geq 0, \tag{6.3}$$

where $\bar m = \bar\rho\bar v$.

Combining the uniform estimates and the compactness of weak entropy dissipation measures in H^{-1}_{loc} with the compensated compactness argument and the reduction of the measure-valued solution $\nu_{t,x}$, we conclude

Theorem 6.1 (Chen-Perepelitsa [45])**.** *Let the initial functions $(\rho_0^\varepsilon, v_0^\varepsilon)$ be smooth and satisfy the following conditions: There exist $E_0, E_1, M_0 > 0$, independent of ε, and $c_0^\varepsilon > 0$ such that*

(i) $\rho_0^\varepsilon(x) \geq c_0^\varepsilon > 0, \quad \int \rho_0^\varepsilon(x)|v_0^\varepsilon(x) - \bar v(x)|\, dx \leq M_0 < \infty$;

(ii) *The total mechanical energy with respect to $(\bar\rho, \bar u)$ is finite:*

$$\int \Big(\frac{1}{2}\rho_0^\varepsilon(x)|v_0^\varepsilon(x) - \bar v(x)|^2 + h(\rho_0^\varepsilon(x), \bar\rho(x))\Big)dx \leq E_0 < \infty,$$

where $h(\rho, \bar\rho) = \rho e(\rho) - \bar\rho e(\bar\rho) - \big(e(\bar\rho) + \bar\rho e'(\bar\rho)\big)(\rho - \bar\rho)$;

(iii) $\varepsilon^2 \int \frac{|\rho^\varepsilon_{0,x}(x)|^2}{\rho^\varepsilon_0(x)^3}\,dx \le E_1 < \infty$;

(iv) $(\rho^\varepsilon_0(x), \rho^\varepsilon_0(x)v^\varepsilon_0(x)) \to (\rho_0(x), \rho_0(x)v_0(x))$ *in the sense of distributions as* $\varepsilon \to 0$, *with* $\rho_0(x) \ge 0$ *a.e.*

Let $(\rho^\varepsilon, m^\varepsilon), m^\varepsilon = \rho^\varepsilon v^\varepsilon$, *be the solution of the Cauchy problem* (6.1)–(6.2) *for the Navier-Stokes equations with initial data* $(\rho^\varepsilon_0(x), \rho^\varepsilon_0(x)v^\varepsilon_0(x))$ *for each fixed* $\varepsilon > 0$. *Then, when* $\varepsilon \to 0$, *there exists a subsequence of* $(\rho^\varepsilon, m^\varepsilon)$ *that converges almost everywhere to a finite-energy entropy solution* (ρ, m) *of the Cauchy problem* (3.13) *and* (6.2) *with initial data* $(\rho_0(x), \rho_0(x)v_0(x))$ *for the isentropic Euler equations with* $\gamma > 1$.

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Gui-Qiang G. Chen, Mathematical Institute, University of Oxford, Oxford, OX1 3LB, UK; and Department of Mathematics, Northwestern University, Evanston, IL 60208, USA

E-mail address: `chengq@maths.ox.ac.uk;gqchen@math.northwestern.edu`